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AN EVALUATION OF THE EIGENVALUE APPROACH FOR DETERMINING THE MEMBERSHIP VALUES IN FUZZY SETS

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Abstract: The membership values of the elements of a fuzzy set are of key importance in any theoretical or practical application of fuzzy set theory. Although there are many methods that evaluate membership values, the method proposed by Saaty [8, 9] based on a matrix of pairwise comparisons and eigenvalue theory, is the backbone of many other methods. In this paper we evaluate the above method by using a forward error analysis approach with the assumption that the true membership values in a fuzzy set are continuous in the interval (0, 1). The results reveal that the eigenvalue method is dramatically inaccurate even for fuzzy sets with few members.

Keywords: Fuzzy set; degree of membership; eigenvector; pairwise comparisons; decision making.

1. Introduction

Membership values are used in order to determine the degree of membership of the elements of a fuzzy set. As the 1 800 references given by Gupta et al. [3] demonstrate, the evaluation of the membership values is of critical importance in applications of fuzzy set theory to Engineering and Science. In most considerations the most representative members in a set are assigned the value of 1.00 while non-members the value of 0.00. Consequently, in-between members are assigned values from the interval (0, 1). Lakoff [7] and other psychologists have found that although people can easily identify the representative members in a fuzzy set, they have difficulties in identifying the in-between members. A reciprocal matrix of pairwise comparisons, as proposed by Saaty [8, 9], has received considerable attention by many researchers in the field. Many attempts to evaluate membership values use the above matrix as the input data (see, for example, [1, 2, 4, 5, 6, 8, 9, 11]). Saaty has suggested a way of evaluating membership values that is based on eigenvalue theory. If the input data are consistent

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enough (as explained in the next section) Saaty claims that the eigenvalue approach is accurate. However, the following evaluation reveals that the eigenvalue approach yields extremely high failure rates.

2. Background information

Let $A_1, A_2, \ldots, A_n$ be the members of a fuzzy set. We are interested in evaluating the membership values of the above members. Saaty [8, 9] proposes to use a matrix A of rational numbers taken from the set $\{\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \ldots, 1, 2, \ldots, 7, 8, 9\}$. Each entry of the above matrix A represents a pairwise judgment. Specifically, the entry a_{ij} denotes the number that estimates the relative membership of element A_i when it is compared with element A_j. Obviously, $a_{ij} = 1/a_{ji}$ and $a_{ii} = 1$. That is, the matrix is a reciprocal one.

Let us first examine the case in which it is possible to have perfect values a_{ij}. In this case it is $a_{ij} = W_i/W_j$ (W_s denotes the actual value of element s) and the previous reciprocal matrix A is consistent. That is,

$$a_{ij} = a_{ik}a_{kj} \quad (i, j, k = 1, 2, 3, \ldots, n, \text{ where } n \text{ is the number of elements in the fuzzy set}). \tag{1}$$

It can be proved that A has rank 1 with $\lambda = n$ its nonzero eigenvalue. Then we have

$$Ax = nx \quad \text{where } x \text{ is an eigenvector.} \tag{2}$$

From the fact that $a_{ij} = W_i/W_j$ the following are obtained:

$$\sum_{j=1}^{n} a_{ij}W_j = \sum_{j=1}^{n} W_i = nW_i, \quad i = 1, 2, 3, \ldots, n, \tag{3}$$

or

$$AW = nW. \tag{4}$$

Equation (4) states that n is an eigenvalue of A with W a corresponding eigenvector. The same equation also states that in the perfectly consistent case (i.e., $a_{ij} = a_{ik}a_{kj}$) the vector W, with membership values from the elements $1, 2, 3, \ldots, n$, is the principal right-eigenvector (after normalization) of matrix A.

In the non-consistent case (which is the most common) the pairwise comparisons are not perfect, that is, the entry a_{ij} might deviate from the real ratio W_i/W_j (i.e., from the ratio of the real membership values W_i and W_j). In this case, the previous expression (1) does not hold for all the possible combinations. Now the new matrix A can be considered as a perturbation of the previous consistent case. When the entries a_{ij} change slightly then the eigenvalues change in a similar fashion [9]. Moreover, the maximum eigenvalue is close to n (greater than n) while the remaining eigenvalues are close to zero. Thus, in order to find the membership values in the non-consistent cases, one should find an eigenvector that corresponds to the maximum eigenvalue $\lambda_{\max}$. That is to say, one must find the principal right-eigenvector W that satisfies

$$AW = \lambda_{\max}W \quad \text{where} \quad \lambda_{\max} \approx n.$$

Table 1. Random Consistency index (RC)

n	1	2	3	4	5	6	7	8	9
RC	0	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45

Saaty estimates the principal right-eigenvector W by multiplying the entries in each row of matrix A together and taking the n-th root (n is the number of the elements in the fuzzy set). Since we desire to have values that add up to 1.00 we normalize the previously found vector. If we want to have the element with the highest value to have membership value equal to 1.00 we divide the previously found vector by the highest value.

Under the assumption of total consistency, if the judgments are Gamma distributed (something that Saaty assumes), the principal right-eigenvector of the resultant reciprocal matrix A is Dirichlet distributed. If the assumption of total consistency is relaxed, then Vargas [14] shows that the hypothesis that the principal right-eigenvector follows a Dirichlet distribution is accepted if the consistency ratio is 0.10 or less.

The consistency ratio (CR) is obtained by first estimating $\lambda_{\max}$. Saaty estimates $\lambda_{\max}$ by adding the columns of matrix A and then multiplying the resulting vector with the vector W. Then he uses what he calls the consistency index (CI) of the matrix A. He defines CI as follows:

$$\mathrm{CI} = (\lambda_{\max} - n)/(n-1).$$

Then, the consistency ratio CR is obtained by dividing the CI by the Random Consistency index (RC) as given in Table 1.

Each RC is an average random consistency index derived from a sample of size 500 of randomly generated reciprocal matrices with entries from the set $\{\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \ldots, 1, 2, \ldots, 7, 8, 9\}$ to see if its CI is 0.10 or less. If the previous approach yields a CR greater than 0.10 then a re-examination of the pairwise judgments is recommended until a CR less than or equal to 0.10 is achieved.

3. The concept of the closest discrete pairwise matrix

The following forward error analysis is based on the assumption that in the real world the membership values in a fuzzy set take on continuous values. This assumption is believed to be a reasonable one since it captures the majority of real world cases.

Let $\omega_1, \omega_2, \omega_3, \ldots, \omega_n$ be the real (and thus unknown) membership values of a fuzzy set with n members. If the decision maker knew the above real values then he would be able to have constructed a matrix with the real pairwise comparisons. In this matrix, say matrix A, the entries are $\alpha_{ij} = \omega_i/\omega_j$. This matrix is called the Real Continuous Pairwise matrix, or the RCP matrix. Since in the real world the ω_i's are unknown so are the entries α_{ij} of the previous matrix. However, we can assume here that the decision maker is able to determine the

Applying the eigenvalue method the following membership values result:

$$w_1 = 0.60000, \quad w_2 = 0.20000, \quad \text{and} \quad w_3 = 0.20000.$$

and the corresponding ranking is $R_1 = 1$, R_2, and $R_3 = 2$.

Obviously, these results contradict the real membership values and ranking of the members of the fuzzy set of this example.

4. Analysis of the failure rates yielded by the eigenvalue approach

In order for the failure rates of the eigenvalue method to be determined, random problems of different sizes were generated and tested as in the previous example. For each such test real membership values were generated randomly from the continuous interval $(0, 1)$. However, because the Saaty matrices use values from the set $\{\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \ldots, 1, 2, \ldots, 7, 8, 9\}$ only the random problems that have RCP matrices with entries within the continuous interval $[\frac{1}{9}, \frac{9}{1}]$ were considered. The RCP matrix, with the real pairwise comparisons was constructed, and after that the CDP matrix was determined. Then the eigenvalue approach was applied. Any contradictions of the rankings were recorded as a failure. This simulation program was written in FORTRAN with the appropriate IMSL subroutines. The failure rates for fuzzy sets of order 3, 4, 5, . . . , 30 are presented in Table 2 and depicted in Figure 1. The lower line in Figure 1 represents the average number of rank inversions. A rank inversion occurs when the eigenvalue method inverses the real ranking of some members in a fuzzy set. The top line represents the total number of failures. That is, both the number of rank inversions and the number of failures to distinguish between members that in reality are different (as in the example).

The findings reveal that the failure rate for the eigenvalue method increases

Table 2. Failure rates of the eigenvalue method for fuzzy sets of different order (results are based on one thousand observations)

Order of set	Inversion rate	Total failure rate	Order of set	Inversion rate	Total failure rate
3	0.04	52.3	17	69.5	98.7
4	0.11	58.3	18	78.3	99.6
5	0.34	68.7	19	81.7	99.4
6	0.68	74.2	20	85.5	99.8
7	10.4	78.5	21	87.3	99.7
8	14.8	85.0	22	91.4	99.8
9	19.4	90.3	23	93.0	100.0
10	28.9	90.7	24	95.2	99.8
11	34.0	93.0	25	96.4	100.0
12	42.4	94.9	26	96.5	100.0
13	45.6	95.3	27	97.3	99.9
14	56.0	96.9	28	98.7	100.0
15	61.4	98.1	29	99.0	100.0
16	68.3	99.0	30	99.4	100.0

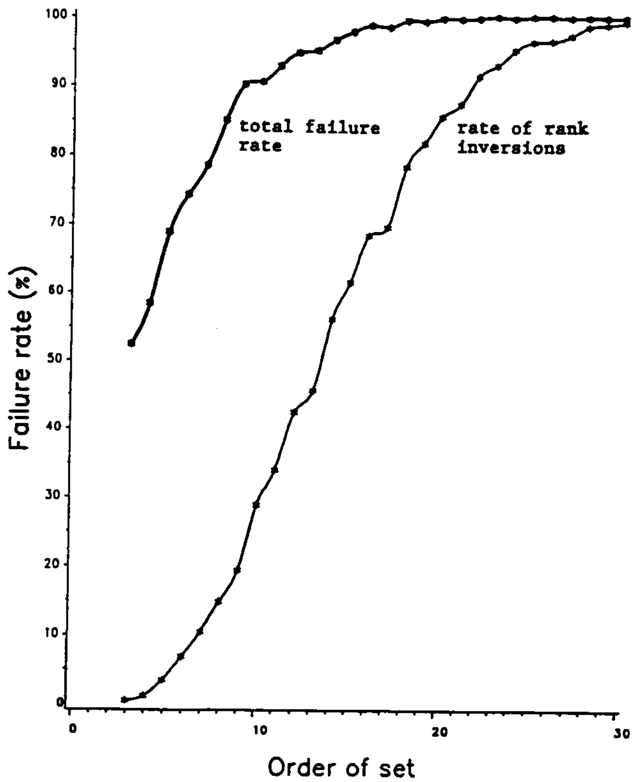

Fig. 1. Failure rates of the eigenvalue method for fuzzy sets of different order (results are based on one thousand observations).

dramatically with the number of members of the fuzzy set. Although the rate of rank inversions is relatively low for small fuzzy sets, it increases very fast with the size of a set. More specifically, sets with more than 14 members cause the eigenvalue method to have rate of rank inversions more than 50%. The total failure rate is dramatically high even for small sets (see Table 2 and Figure 1).

An examination of the conditions under which the eigenvalue method fails reveals that failures occur when two or more real membership values are close to each other. The mapping of the RCP matrix onto the corresponding CDP matrix fails to reflect these small differences and a failure of the eigenvalue method occurs. Since in larger sets it is more likely that some membership values will be close to each other, the eigenvalue method fails more dramatically as the order of the set increases.

5. Conclusions

The eigenvalue approach, as proposed by Saaty, has captured the interest of many researchers. Its main advantages are the nice mathematical properties of the method and the fact that the input data are rather easy to obtain.

The forward error analysis in this paper reveals that this approach yields high failure rates. The present findings are based on the very favorable assumption that the decision maker deals with the Closest Discrete Pairwise matrix for each problem. However, this phenomenon rarely occurs in real world situations. More specifically, the more members a fuzzy set has, the more likely it is that the pairwise matrix the decision maker derives is significantly different than the CDP matrix. This fact indicates that in actual real world situations the eigenvalue approach degenerates even more rapidly than what is found in this paper. As it is shown by Triantaphyllou et al. [10, 12] small changes in the membership values or even repetitions of identical alternatives can mean the difference between selecting one alternative over another in many decision-making problems. Since accurate evaluation of membership values is crucial to many engineering and scientific problems, further research in this subject is critical.

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