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Linear Programming Based Decomposition Approach in Evaluating Priorities from Pairwise Comparisons and Error Analysis[1]

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Abstract. One of the most difficult issues in many real-life decision-making problems is how to estimate the pertinent data. An approach which uses pairwise comparisons was proposed by Saaty and is widely accepted as an effective way of determining these data. Suppose that two matrices with pairwise comparisons are available. Furthermore, suppose that there is an overlapping of the elements compared in these two matrices. The problem examined in this paper is how to combine the comparisons of the two matrices in order to derive the priorities of the elements considered in both matrices. A simple approach and a linear programming approach are formulated and analyzed in solving this problem. Computational results suggest that the LP approach, under certain conditions, is an effective way for dealing with this problem. The proposed approach is of critical importance because it can also result in a reduction of the total required number of comparisons.

Key Words. Pairwise comparisons, eigenvectors, analytic hierarchy process, linear programming, fuzzy sets, membership values, artificial intelligence.

1. Introduction

For a long time, it has been recognized that an exact description of many real physical situations is virtually impossible. This is due to the high

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degree of imprecision involved in real-world situations. Zadeh (Refs. 1 and 2) proposed fuzzy set theory as the means for quantifying the inherent fuzziness that is present in ill-posed problems. Fuzziness is a type of imprecision which may be associated with sets in which there is no sharp transition from membership to nonmembership (Ref. 3). Examples of fuzzy sets are classes of objects characterized by such adjectives as large, small, serious, simple, approximate, etc. (Ref. 3).

A comprehensive description of the importance of fuzzy set theory in engineering and scientific problems is best illustrated in the more than 1,800 references given in Refs. 4–12. Currently, an increasingly large number of researchers has been faced with the problem that either their data or their background knowledge is fuzzy. It is particularly critical to people building expert systems and decision support systems, for the knowledge they are dealing with is almost always riddled with vague concepts and judgmental rules (e.g., Refs. 13–17). Some recent developments of fuzzy theory on decision-making problems are reported in Ref. 18. The most critical step in any application of fuzzy set theory is to effectively estimate the pertinent membership values. Although this is a fundamental problem, there is not a unique way of determining membership values in a fuzzy set. This is mainly due to the way different researchers perceive this problem.

A method proposed by Saaty (Refs. 19–21), which is based on pairwise comparisons, has captured the interest of many researchers (see, for example, Refs. 22–25). According to this approach, the decision maker needs to compare the elements of a fuzzy set by considering all possible pairs. That is, if there are N members then he needs to perform $N \times (N-1)/2$ pairwise comparisons. Each comparison reflects the personal judgment of the decision maker on how strongly one member of a set belongs to that set when it is compared with another member of the same set.

Saaty (Ref. 20) proposes the use of the scale depicted in Table 1 as the means for quantifying these pairwise comparisons. Other researchers have proposed alternative scales (e.g. Refs. 16 and 27). A scale is nothing but a mapping between a set of discrete linguistic phrases, such as "A is more important than B", and a set of numerical intensities. In Ref. 28, two classes of a total of 78 scales were studied in terms of two evaluative criteria. In that study, it was found that no single scale is the best, but different scales are more effective under different circumstances. However, the original Saaty scale is the most widely used, and thus this is the one used in the computational experiments described in this paper.

After the pairwise comparisons on N entities are quantified by using a scale, they are used to form the entries of a reciprocal matrix of order N. The (i, j) entry of this matrix reflects the pairwise comparison of the ith

Table 1. Scale of relative importances, according to Saaty (Ref. 21).

Intensity of importance	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective.
3	Weak importance of one over another	Experience and judgment slightly favor one activity over another.
5	Essential or strong importance	Experience and judgment strongly favor one activity over another.
7	Demonstrated importance	An activity is strongly favored and its dominance demonstrated in practice.
9	Absolute importance	Evidence favoring one activity over another is of the highest possible order of affirmation.
2, 4, 6, 8	Intermediate values between the two adjacent judgments	When compromise is needed.
Reciprocals of above nonzero	If activity i has one of the above nonzero numbers assigned to it when compared with activity j, then j has the reciprocal value when compared with i.	

entity when it is compared with the jth entity. Apparently, entry $a_{ii} = 1$ and $a_{ij} = 1/a_{ij}$. Ideally, if the decision maker is perfectly consistent, then the following relation should also be true:

$$a_{ij} = a_{ik} \times a_{jk}, \qquad i, j, k = 1, 2, 3, \ldots, N.$$

In real-life situations, there is no guarantee that the above matrix will be consistent. Saaty uses a measurement of consistency, called the consistency index (CI), to express the consistency of the comparisons. If the CI coefficient is not within certain acceptable limits, then the comparisons have to be repeated until an acceptable consistency level is reached (Ref. 29). After the above reciprocal matrix has been established, the principal right eigenvector of this matrix is estimated in order to express the relative priority of the N entities (Ref. 21). That is, according to Saaty (Ref. 21), the entries of this eigenvector reflect the membership values of the members in the set under consideration. An evaluation of the effectiveness of this eigenvector approach can be found in Ref. 30.

Other researchers have proposed the use of a logarithmic regression model (e.g., Refs. 26, 27, 31, 32) instead of the eigenvector approach. In Ref. 33, a least squares approach is proposed and compared with some other related methods. However, the eigenvector approach is simple and widely used in the literature. Therefore, this is the method used in the computational experiments described in this paper.

The main problem examined in this paper is how to estimate the priorities (i.e., membership values) of N elements when the pairwise comparisons of two subsets with N_1 and N_2 elements, where $N_1 + N_2 > N$, are known. Solving this problem is of critical importance for a number of reasons. When the number of elements to be processed is large, the number of all possible comparisons is very large. For instance, for a collection of 20 elements one needs to perform $190 = 20 \times 19/2$ comparisons. Therefore, finding a way for reducing the total number of comparisons is of great practical importance. One may want to partition a large collection of elements into a number of smaller groups. The elements may be clustered into these groups by placing very similar elements in the same group. In this way, the decision maker will evaluate elements which are more homogeneous. This has the potential of deriving more accurate comparisons than when one has to compare very different elements.

Another application comes from the area of performing the union operation on the membership values of two sets. If the pairwise comparisons used to derive these membership values are available, then one may want to utilize them in order to determine the membership values of the union of the two sets. This operation can be applicable when performing the union operation in fuzzy data bases.

A similar problem was examined by Harker in Ref. 34. In that approach, the decision maker starts with a minimum set of N comparisons, where N is the number of elements. Then, an expert system-like approach is developed for determining what should be the next comparisons to be made. In this way, the decision maker determines the comparisons in a guided manner. The main difference of the problem examined in this paper is that the pairwise comparisons are assumed to be clustered into two groups of N_1 and N_2 elements each, where $N_1 + N_2 > N$. Therefore, in the present investigation, we assume that we have two complete collections of pairwise comparisons. The first collection has $N_1(N_1 - 1)/2$ comparisons and the second has $N_2(N_2 - 1)/2$ comparisons.

In this paper, two methods are developed for solving the problem of estimating the priorities from two collections of comparisons. The first method is a simple and straightforward approach, while the second is more sophisticated and uses a linear programming (LP) formulation. The LP approach estimates the missing comparisons of the reciprocal matrix by attempting to minimize the CI coefficient of the matrix defined on all the N elements. The two methods are also evaluated in terms of a forward error analysis. The computational results reveal that, when the number of common pairwise comparisons is high enough (i.e., $N_1 + N_2$ is significantly greater than N), then the LP approach is the most reliable approach.

2. Problem Description

The main problem examined in this paper is best described in terms of an example. Suppose that there are five entities (say A_1, A_2, A_3, A_4, A_5) for which a decision maker wishes to find their priorities by using pairwise comparisons. These entities may be the alternatives of a problem, and the decision maker wishes to find their relative priorities in terms of a single decision criterion. Furthermore, suppose that the decision maker has available the pairwise comparisons when these five entities are grouped into two subsets of four and three members as follows: The first subset has the entities $\{A_1, A_2, A_3, A_4\}$, and the second subset is $\{A_3, A_4, A_5\}$.

Let the reciprocal matrices with the pairwise comparisons for the previous two subsets be as follows:

$$M_1 = \begin{bmatrix} 1.000 & 2.000 & 1.000 & 0.333 \\ 0.500 & 1.000 & 0.500 & 0.200 \\ 1.000 & 2.000 & 1.000 & 0.333 \\ 3.000 & 5.000 & 3.000 & 1.000 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 1.000 & 0.333 & 0.500 \\ 3.000 & 1.000 & 2.000 \\ 2.000 & 0.500 & 1.000 \end{bmatrix}.$$

In this paper, it is always assumed that the pairwise comparisons of the same pair of entities is the same regardless of the matrix. This is the reason why the comparison between A_3 and A_4 is the same (equal to 0.3333) in both the M_1 and M_2 matrices.

It can be verified that both of the previous matrices are satisfactorily consistent (their CI values are less than 0.10) and thus can be used to derive the relative priorities of the two groups of elements. Saaty suggests in Ref. 19 that an effective way for estimating the principal right eigenvector for this type of matrices is to first calculate the geometric means of each row and then normalize these means. When this procedure is applied to the previous two matrices M_1 and M_2, the following two vectors P_1 and P_2 with the relative priorites are derived, respectively:

$$P_1 = \begin{bmatrix} 0.1855 \\ 0.9710 \\ 0.1855 \\ 0.5318 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.1634 \\ 0.5396 \\ 0.2970 \end{bmatrix}$$

Observe that, from the first vector, the ratio of the relative priorities of the elements A_3 and A_4 is $0.3488 = 0.1855/0.5318$; from the second vector, the same ratio is $0.3028 = 0.1634/0.5396$. Therefore, the question that we seek to answer in this example is: what are the relative priorities when all the five elements are considered together?

We can view the previous two matrices as parts of a larger matrix which is defined on the entire five elements. When the matrices M_1 and M_2 are combined, the following 5×5 matrix M is derived, where the asterisk indicates undetermined comparisons:

$$M = \begin{bmatrix} 1.000 & 2.000 & 1.000 & 0.333 & * \\ 0.500 & 1.000 & 0.500 & 0.200 & * \\ 1.000 & 2.000 & 1.000 & 0.333 & 0.500 \\ 3.000 & 5.000 & 3.000 & 1.000 & 2.000 \\ * & * & 2.000 & 0.500 & 1.000 \end{bmatrix}.$$

In other words, only the comparisons A_1/A_5 and A_2/A_5 are missing.

In general, suppose that there are N entities divided into two subsets of N_1 and N_2 elements each, where $N_1 + N_2 > N$. Without loss of generality, suppose that the $K = N_1 + N_2 - N$ last elements in the first subset correspond, with the same order, to the first K elements in the second subset (as was the case in the previous numerical example with $K = 2 = 4 + 3 - 5$).

When we consider the matrix with all possible comparisons for the N elements, from the above considerations it follows that the decision maker has available the following comparisons:

$$a_{ij}, \qquad i, j = 1, 2, 3, \ldots, N_1,$$

$$a_{ij}, \qquad i, j = N_2, N_2 + 1, N_2 + 1, \ldots, N.$$

The comparisons

$$a_{ij}, \qquad i = N_1 + 1, N_1 + 2, \ldots, N,$$

$$j = 1, 2, 3, \ldots, N_2 - 1,$$

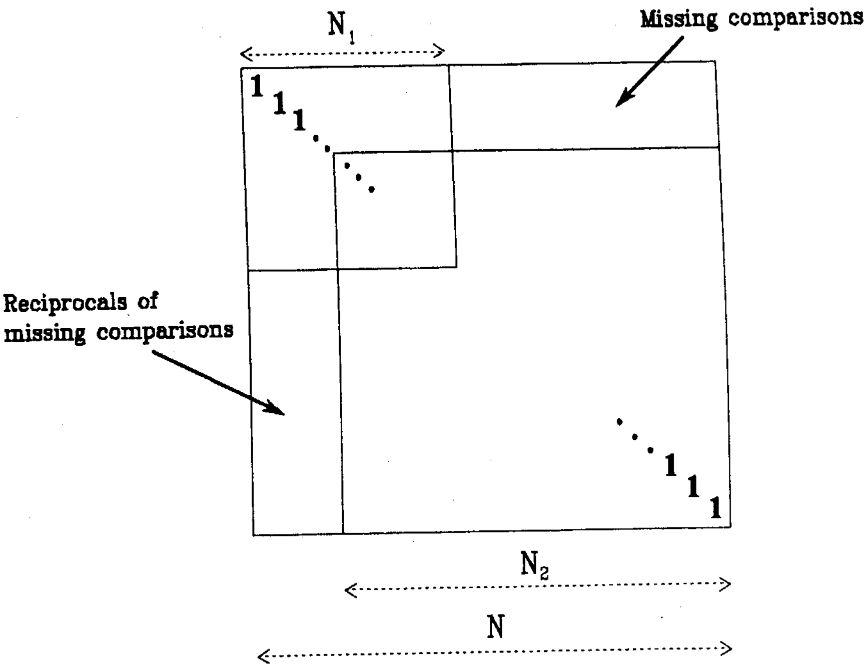

Fig. 1. Partitioning of the $N \times (N-1)/2$ pairwise comparisons.

are undetermined (recall that $a_{ji} = 1/a_{ij}$). In other words, the $N \times N$ matrix with the $N \times (N-1)/2$ comparisons can be viewed to be partitioned into submatrices as depicted in Fig. 1.

In the next section, two procedures are developed for estimating the missing comparisons. The first procedure is a simple and straightforward one, while the second procedure attempts an error minimization and is based on an LP formulation.

3. Solution Approaches

3.1. Simple Approach. As was mentioned in the introduction section, perfectly consistent reciprocal matrices with pairwise comparisons satisfy the following relationship:

$$a_{ij} = a_{ik} \times a_{jk}, \qquad i, j, k = 1, 2, 3, \ldots, N. \tag{1}$$

From relationship (1) it follows that, in the perfectly consistent case, the missing comparisons, denoted here as X_{ij} can be determined as follows (see also Fig. 1):

$$X_{ij} = a_{ik}/a_{jk}, \qquad i = 1, 2, 3, \ldots, N - N_2, \quad j = N_1 + 1, N_2 + 1, N_1 + 3, \ldots, N, \quad k = N - N_2 + 1, \ldots, N_1. \tag{2}$$

In nonconsistent cases, the previous relationship (2) is not always true. However, the unknown terms X_{ij} are expected to be as close to the products $a_{ik} \times a_{jk}$ as possible. Therefore, it is reasonable to determine the unknown terms X_{ij} as the averages of all possible products. In other words, a simple way is to calculate the terms X_{ij} as follows:

$$X_{ij} = \left[\sum_{k=N-N_2+1}^{N_1} (a_{ik} / a_{jk}) \right] / (N_1 + N_2 - N),$$

$$i = 1, 2, 3, \ldots, N - N_2,$$

$$j = N_1 + 1, N_1 + 2, N_1 + 3, \ldots, N. \tag{3}$$

After these averages are calculated, the missing entries of the entire matrix are estimated. Next, the eigenvector approach or any other pertinent approach can be applied on the complete matrix, and thus the final priorities of the N entities can be estimated.

Although the above averages can be calculated in a straightforward manner, the above approach fails to capture the requirement for the following relationship (4):

$$X_{ij} \approx (a_{ii'} \times a_{j'j}) \times X_{i'j'},$$

$$i, i' = 1, 2, 3, \ldots, N - N_2,$$

$$j, j' = N_1 + 1, N_1 + 2, N_1 + 3, \ldots, N. \tag{4}$$

Note that relationship (4) follows directly from the fact that $a_{ij} = 1/a_{ji}$, for any $i, j = 1, 2, 3, \ldots, N$.

3.2. Linear Programming Approach. The previous approach can be modified and transformed into a more sophisticated procedure. Consider relationship (3). When one wishes to incorporate relationship (4), then relationship (3) may not hold as an equality but instead will be satisfied as follows:

$$X_{ij} \approx \left[\sum_{k=N-N_2+1}^{N_1} (a_{ik} / a_{jk}) \right] / (N_1 + N_2 - N),$$

$$i = 1, 2, 3, \ldots, N - N_2,$$

$$j = N_1 + 1, N_1 + 2, N_1 + 3, \ldots, N. \tag{5}$$

That is, now the left-hand side is approximately equal to the right-hand side. If we wish relationship (5) to be an equality, then an error term, denoted as e_{ij}, can be introduced as follows:

$$X_{ij}=\left[\sum_{k=N-N_2+1}^{N_1}(a_{ik}/a_{jk})\right]/(N_1+N_2-N)+e_{ij},$$
$$i=1,2,3,\dots,N-N_2,$$
$$j=N_1+1,N_1+2,N_1+3,\dots,N. \tag{6}$$

Similarly, relationship (4) can be transformed into an equality by introducing an error term, denoted as $e_{ij}^{i'j'}$, as follows:

$$X_{ij}=(a_{ii'}\times a_{j'j})\times X_{i'j'}+e_{ij}^{i'j'}, \quad i,i'=1,2,3,\dots,N-N_2,$$
$$j,j'=N_1+1,N_1+2,N_1+3,\dots,N. \tag{7}$$

Relationships (6) and (7) suggest that a reasonable treatment to the problem of estimating the missing entries (and thus determining the final priorities) is to attempt to minimize the sum of all the previous error terms. This is in accordance with the implicit assumption that the decision maker attempts to be as consistent in his judgments as possible. Since the errors may be positive or negative, we would like to minimize the sum of their absolute values. Thus, this consideration leads to the following LP formulation:

$$\min \quad f=\sum_{i=1}^{N-N_2}\sum_{j=N_1+1}^{N}|e_{ij}|+\sum_{i=1}^{N-N_2}\sum_{j=N_1+1}^{N}\sum_{i'=1}^{N-N_2}\sum_{j'=N_1+1}^{N}|e_{ij}^{i'j'}|,$$

$$\text{s.t.} \quad X_{ij}=\left[\sum_{k=N-N_2+1}^{N_1}(a_{ik}/a_{jk})\right]/(N_1+N_2-N)+e_{ij},$$
$$i=1,2,3,\dots,N-N_2,$$
$$j=N_1+1,N_1+2,N_1+3,\dots,N,$$
$$X_{ij}=(a_{ii'}\times a_{j'j})\times X_{i'j'}+e_{ij}^{i'j'},$$
$$i,i'=1,2,3,\dots,N-N_2$$
$$j,j'=N_1+1,N_1+2,N_1+3,\dots,N,$$

the variables X_{ij} are ≥ 0,

the variables e_{ij} and $e_{ij}^{i'j'}$ are unrestricted.

The absolute values in the previous LP model can be eliminated by introducing the following transformations in the body of constraints:

$$e_{ij}=p_{ij}-n_{ij}, \quad i=1,2,3,\dots,N-N_2,$$
$$j=N_1+1,N_1+2,N_1+3,\dots,N,$$
$$e_{ij}^{i'j'}=P_{ij}^{i'j'}-N_{ij}^{i'j'}, \quad i,i'=1,2,3,\dots,N-N_2$$
$$j,j'=N_1+1,N_1+2,N_1+3,\dots,N,$$

the variables $p_{ij}, n_{ij}, P_{ij}^{i'j'}, N_{ij}^{i'j'}$ are ≥ 0, and also modifying the objective function f as follows:

$$f = \sum_{i=1}^{N-N_2} \sum_{j=N_1+1}^{N} [p_{ij} + n_{ij}] + \sum_{i=1}^{N-N_2} \sum_{j=N_1+1}^{N} \sum_{i'=1}^{N-N_2} \sum_{j'=N_1+1}^{N} [P_{ij}^{i'j'} + N_{ij}^{i'j'}].$$

This is true because if say the actual term e_{ij} has to be negative, then it can be easily seen (from the linear dependences in the columns of the new constraints) that the value of the variable p_{ij} will be equal to zero (as a nonbasic variable), while the variable n_{ij} will be greater than zero. In other words,

$$|e_{ij}| = p_{ij} + n_{ij},$$
$$|e_{ij}^{i'j'}| = P_{ij}^{i'j'} + N_{ij}^{i'j'}.$$

Therefore, the previous LP model takes the following form:

$$\min \quad f = \sum_{i=1}^{N-N_2} \sum_{j=N_1+1}^{N} [p_{ij} + n_{ij}] + \sum_{i=1}^{N-N_2} \sum_{j=N_1+1}^{N} \sum_{i'=1}^{N-N_2} \sum_{j'=N_1+1}^{N} [P_{ij}^{i'j'} + N_{ij}^{i'j'}],$$

$$\text{s.t.} \quad X_{ij} = \left[\sum_{k=N-N_2+1}^{N_1} (a_{ik}/a_{jk}) \right] / (N_1 + N_2 - N) + p_{ij} - n_{ij},$$

$i = 1, 2, 3, \ldots, N - N_2$

$j = N_1 + 1, N_1 + 2, N_1 + 3, \ldots, N,$

$X_{ij} = (a_{ii'} \times a_{jj'}) \times X_{i'j'} + P_{ij}^{i'j'} - N_{ij}^{i'j'},$

$i, i' = 1, 2, 3, \ldots, N - N_2$

$j, j' = N_1 + 1, N_1 + 2, N_1 + 3, \ldots, N,$

the variables X_{ij}, p_{ij}, n_{ij} and $P_{ij}^{i'j'}, N_{ij}^{i'j'}$ are ≥ 0.

From the above considerations, it follows that the proposed LP model uses $(N - N_1)(N - N_2)[3 + 2(N - N_1)(N - N_2)]$ continuous variables and $(N - N_1)(N - N_2)[1 + (N - N_1)(N - N_2)]$ constraints. It can also easily be seen that, if the input pairwise comparisons (i.e., the ones defined on the two subgroups of N_1 and N_2 members) are perfectly consistent, then at optimality the value of the objective function of the previous LP problem is equal to zero (i.e., all errors vanish). Moreover, the optimal solution of the X_{ij} variables is given by relationship (3). The previous concepts and issues are further illustrated in the following numerical example.

3.3. Numerical Example. In this example, we use the same data as in the illustrative example described in Section 2. Therefore, the pertinent LP model will have 14 variables and 6 constraints, since $N = 5$, $N_1 = 4$, and $N_2 = 3$. Note that

$$(1/2)(a_{13}/a_{53} + a_{14}/a_{54}) = 0.583,$$

$$(1/2)(a_{23}/a_{53} + a_{24}/a_{54}) = 0.325,$$

$$(a_{11} \times a_{55}) = 1.000,$$

$$(a_{12} \times a_{55}) = 2.000.$$

Therefore, the constraints of the LP formulation for this numerical example are as follows:

$$X_{15} = 0.583 + p_{15} - n_{15},$$

$$X_{25} = 0.325 + p_{25} - n_{25},$$

$$X_{15} = 2.000X_{25} + P_{15}^{25} - N_{15}^{25},$$

$$X_{25} = 0.500X_{15} + P_{25}^{15} - N_{25}^{15},$$

$$X_{15} = 1.000X_{15} + P_{15}^{15} - N_{15}^{15},$$

$$X_{25} = 1.000X_{25} + P_{25}^{25} - N_{25}^{25}.$$

Observe that the last two constraints are redundant, since we can directly set

$$P_{15}^{15} = N_{15}^{15} = P_{25}^{15} = N_{25}^{25} = 0.$$

In general, we can always set

$$P_{ij}^{ij} = N_{ij}^{ij} = 0.$$

This observation suggests that the LP formulation, defined in the previous subsection, can have fewer variables and constraints.

From the previous discussions, it follows that the LP formulation for this example is

$$\min \quad f = p_{15} + n_{15} + p_{25} + n_{25} + P_{15}^{25} + N_{15}^{25} + P_{25}^{15} + N_{25}^{15},$$

$$\text{s.t.} \quad X_{15} - p_{15} + n_{15} = 0.583,$$

$$X_{25} - p_{25} + n_{25} = 0.325,$$

$$X_{15} - 2.000X_{25} - P_{15}^{25} + N_{15}^{25} = 0,$$

$$X_{25} - 0.500X_{15} - P_{25}^{15} + N_{25}^{15} = 0,$$

$$X_{15}, X_{25}, p_{15}, p_{25}, n_{15}, n_{25}, P_{15}^{25}, P_{25}^{15}, N_{15}^{25}, N_{25}^{15} \geq 0.$$

An optimal solution to this LP problem is

$$X_{15}^* = 0.5830, \qquad X_{25}^* = 0.2915, \qquad n_{25}^* = 0.0335,$$

and all other variables are equal to zero. The value of the objective function at optimality is $f = 0.0335$.

From this optimal solution, it follows that the complete 5×5 matrix, with all the pairwise comparisons, is as follows:

$$M' = \begin{bmatrix} 1.000 & 2.000 & 1.000 & 0.333 & 0.583 \\ 0.500 & 1.000 & 0.500 & 0.200 & 0.292 \\ 1.000 & 2.000 & 1.000 & 0.333 & 0.500 \\ 3.000 & 5.000 & 3.000 & 1.000 & 2.000 \\ 1.715 & 3.431 & 2.000 & 0.500 & 1.000 \end{bmatrix}.$$

When the eigenvector approach is applied on the previous pairwise comparisons, the following priorities are derived for the entire set of the five elements A_1, A_2, A_3, A_4, A_5:

$$P' = \begin{bmatrix} 0.1392 \\ 0.0722 \\ 0.1350 \\ 0.4137 \\ 0.2398 \end{bmatrix}.$$

If the non-LP approach is used, then it can be easily verified that

$$X_{15} = 0.583, \qquad X_{25} = 0.325.$$

Therefore, M'' and P'', the matrix with the pairwise comparisons and vector with the priorities, respectively, are as follows:

$$M'' = \begin{bmatrix} 1.000 & 2.000 & 1.000 & 0.333 & 0.583 \\ 0.500 & 1.000 & 0.500 & 0.200 & 0.325 \\ 1.000 & 2.000 & 1.000 & 0.333 & 0.500 \\ 3.000 & 5.000 & 3.000 & 1.000 & 2.000 \\ 1.715 & 3.077 & 2.000 & 0.500 & 1.000 \end{bmatrix},$$

$$P'' = \begin{bmatrix} 0.1478 \\ 0.0783 \\ 0.1433 \\ 0.4390 \\ 0.1916 \end{bmatrix}.$$

4. Computational Experiments

One challenging issue in designing computational experiments is how to generate the pertinent data. In this study, in deriving the pertinent data, we follow a similar strategy as in the experiments reported in Refs. 33, 35, 36, 37, 38, 39.

The following forward error analysis is based on the assumption that, in the real world, the actual relative priorities of the members of a collection of entities take on continuous values. These relative priorities can also be viewed as the degree of belonging to a fuzzy set. This continuity assumption is believed to be reasonable, since it captures the majority of real-world cases. As was mentioned earlier, these members may be a set of alternatives, and the membership values are the degrees that these alternatives meet a single decision criterion.

Let $\omega_1, \omega_2, \omega_3, \ldots, \omega_n$ be the real and thus unknown membership values of a fuzzy set with n members. If the decision maker knew the above real values, then he would be able to have constructed a matrix with the real pairwise comparisons. In this matrix, say matrix A, the entries are $\alpha_{ij} = \omega_i/\omega_j$. This matrix is called the real continuous pairwise (RCP) matrix.

Since in the real world the ω_i's are unknown, so are the entries α_{ij} of the previous matrix. However, we can assume here that the decision maker instead of an unknown entry α is able to determine the closest values taken from the set $\{1/9, 1/8, \ldots, 1, 2, \ldots, 8, 9\}$, if the original Saaty scale is to be used. That is, instead of the real (and thus unknown) value α, one is able to determine a_{ij} such that

$$\text{the difference } |\alpha - a_{ij}| \text{ is minimum,}$$

$$a_{ij} \in \{1/9, 1/8, 1/7, \ldots, 1, 2, \ldots, 7, 8, 9\}.$$

In other words, it is assumed here that one's judgment about the value of the pairwise comparison of the ith element when it is compared with the jth one, is so accurate that, in real life, it is the closest (in absolute value terms) to the values one is supposed to choose from.

The matrix with the a_{ij} entries, which we assume that the decision maker is able to construct, has entries from the discrete and finite set $\{1/9, 1/8, \ldots, 1, 2, \ldots, 8, 9\}$. This second matrix is called the closest discrete pairwise (CDP) matrix. More on some interesting properties of RCP and CDP matrices can be found in Ref. 28.

For illustrative purposes, suppose that the actual relative priorities of a set of five elements, denoted as $\{A_1, A_2, A_3, A_4, A_5\}$, are as follows:

$$\Omega = \begin{bmatrix} 0.1328 \\ 0.0745 \\ 0.1542 \\ 0.3888 \\ 0.2498 \end{bmatrix}.$$

Using the previous data, the corresponding RCP matrix can be found to be as follows:

$$\mathrm{RCP} = \begin{bmatrix} 1.000 & 1.783 & 0.861 & 0.342 & 0.532 \\ 0.561 & 1.000 & 0.483 & 0.192 & 0.298 \\ 1.161 & 2.070 & 1.000 & 0.397 & 0.617 \\ 2.928 & 5.220 & 2.521 & 1.000 & 1.556 \\ 1.881 & 3.354 & 1.620 & 0.643 & 1.000 \end{bmatrix}.$$

This is true because say the element (1, 2) is equal to $1.783 = \omega_1/\omega_2 = 0.1328/0.0745$. A similar interpretation holds for the remaining entries in this RCP matrix. Given the previous RCP matrix, it can be easily verified that the corresponding CDP matrix is

$$\mathrm{CDP} = \begin{bmatrix} 1.000 & 2.000 & 1.000 & 0.333 & 0.500 \\ 0.500 & 1.000 & 0.500 & 0.200 & 0.333 \\ 1.000 & 2.000 & 1.000 & 0.333 & 0.500 \\ 3.000 & 5.000 & 3.000 & 1.000 & 2.000 \\ 2.000 & 3.000 & 2.000 & 0.500 & 1.000 \end{bmatrix}.$$

This is true because say the element (1, 2) is equal to 2.000, since the value 2.000 (taken from the current scale on use) is the closest value to the corresponding entry in the RCP matrix (i.e., 1.783). A similar interpretation holds for the remaining entries in this CDP matrix. Given the previous CDP matrix, the relative priorities derived by using the eigenvector approach are

$$P = \begin{bmatrix} 0.1352 \\ 0.0743 \\ 0.1352 \\ 0.4143 \\ 0.2410 \end{bmatrix}.$$

Next, consider the case in which $N_1 = 4$ and $N_2 = 3$. This setting, along with the previous CDP matrix, creates the data considered in the numerical example discussed in Section 3.3. In that example, it was found that the LP approach and the non-LP approach yield the following relative priorities P_1 and P_2, respectively:

$$P_1 = \begin{bmatrix} 0.1392 \\ 0.0722 \\ 0.1350 \\ 0.4137 \\ 0.2398 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.1478 \\ 0.0783 \\ 0.1433 \\ 0.4390 \\ 0.1916 \end{bmatrix}.$$

From the previous two sets of relative priorities, we can observe that the priorities derived by using the LP approach are closer to the priorities when the CDP matrix is considered (i.e., with vector P). However, the ranking of the five elements, when the LP and non-LP approaches are used, is different than the ranking implied by the priorities in P.

The computational experiments were conducted in a similar manner. Sets with $N = 4, 5, 6, \ldots, 13$ members were assumed. For each case, all possible pairs of values of N_1 and N_2 were considered. For instance, for the case with $N = 6$, the (N_1, N_2) pairs were (3, 3), (3, 4), (3, 5), (4, 4), (4, 5), and (5, 5). In this manner, for the previous values of N, a total of 170 different combinations of N, N_1, N_2 values were generated. These values are depicted in the computational results presented in Tables 2a, 2b, 2c; note that the results are sorted in ascending order of the entries of the fifth column.

Table 2a. Computational results.

N_1	N_2	N	Common PCs (%)	Available PCs (%)	Error rate, LP approach	Error rate, non-LP approach
3	3	4	20	83.33	2	2
3	3	5	0	60	15	5
3	3	6	0	40	82	100
3	4	5	12.5	80	7	5
3	4	6	0	60	21	11
3	4	7	0	42.86	88	100
3	5	6	8.33	80	8	10
3	5	7	0	61.9	27	21
3	5	8	0	46.43	94	100
3	6	7	5.88	80.95	5	8
3	6	8	0	64.29	38	27
3	6	9	0	50	92	100
3	7	8	4.35	82.14	21	19
3	7	9	0	66.67	40	25
3	7	10	0	53.33	98	100
3	8	9	3.33	83.33	15	16
3	8	10	0	68.89	48	41

Table 2a. (continued).

N_1	N_2	N	Common PCs (%)	Available PCs (%)	Error rate, LP approach	Error rate, non-LP approach
3	8	11	0	56.36	75	45
3	9	10	2.63	84.44	23	18
3	9	11	0	70.91	57	32
3	9	12	0	59.09	99	100
3	10	11	2.13	85.45	34	27
3	10	12	0	72.73	53	42
3	10	13	0	61.54	99	100
3	11	12	1.75	86.36	33	30
3	11	13	0	74.36	74	48
3	12	13	1.47	87.18	33	35
4	4	5	33.33	90	0	0
4	4	6	9.09	73.33	12	9
4	4	7	0	57.14	21	9
4	4	8	0	42.86	93	100
4	5	6	23.08	86.67	0	2
4	5	7	6.67	71.43	14	15
4	5	8	0	57.14	33	18
4	5	9	0	44.44	94	100
4	6	7	16.67	85.71	3	10
4	6	8	5	71.43	28	18
4	6	9	0	58.33	47	33
4	6	10	0	46.67	100	100
4	7	8	12.5	85.71	7	5
4	7	9	3.85	72.22	32	21
4	7	10	0	60	58	40
4	7	11	0	49.09	98	100
4	8	9	9.68	86.11	13	16
4	8	10	3.03	73.33	23	25
4	8	11	0	61.82	72	42
4	8	12	0	51.52	100	100
4	9	10	7.69	86.67	5	13
4	9	11	2.44	74.55	47	35
4	9	12	0	63.64	76	49

For each such case, 100 random problems were generated and tested to see if the ranking derived by the non-LP and the LP approaches were identical with the ranking derived when the CDP matrix is processed with the eigenvalue method. Note that sometimes the LP formulation with $N > 13$ required excessive CPU time (when 100 random test problems had to be solved), and thus values of $N > 13$ were dropped from further

Table 2b. Computational results.

N_1	N_2	N	Common PCs (%)	Available PCs (%)	Error rate, LP approach	Error rate, non-LP approach
4	9	13	0	53.85	100	100
4	10	11	6.25	87.27	18	23
4	10	12	2	75.76	51	42
4	10	13	0	65.38	88	61
4	11	12	5.17	87.88	24	26
4	11	13	1.67	76.92	55	44
4	12	13	4.35	88.46	11	19
5	5	6	42.86	93.33	0	1
5	5	7	17.65	80.95	9	11
5	5	8	5.26	67.86	24	16
5	5	9	0	55.56	56	28
5	5	10	0	44.44	98	100
5	6	7	31.58	90.48	4	2
5	6	8	13.64	78.57	12	18
5	6	9	4.17	66.67	38	28
5	6	10	0	55.56	59	35
5	6	11	0	45.45	100	100
5	7	8	24	89.29	4	4
5	7	9	10.71	77.78	18	16
5	7	10	3.33	66.67	35	32
5	7	11	0	56.36	98	100
5	7	12	0	46.97	89	79
5	8	9	18.75	88.89	6	8
5	8	10	8.57	77.78	26	36
5	8	11	2.7	67.27	53	45
5	8	12	0	57.58	84	62
5	8	13	0	48.72	90	75
5	9	10	15	88.89	11	14
5	9	11	6.98	78.18	24	29
5	9	12	2.22	68.18	57	45
5	9	13	0	58.97	95	66
5	10	11	12.24	89.09	11	24
5	10	12	5.77	78.79	43	27
5	10	13	1.85	69.23	55	49
5	11	12	10.17	89.39	13	18
5	11	13	4.84	79.49	43	50
5	12	13	8.57	89.74	18	21
6	6	7	50	95.24	0	0
6	6	8	25	85.71	3	11
6	6	9	11.11	75	19	30
6	6	10	3.45	64.44	44	34
6	6	11	0	54.55	79	51

Table 2b. (continued).

N_1	N_2	N	Common PCs (%)	Available PCs (%)	Error rate, LP approach	Error rate, non-LP approach
6	6	12	0	45.45	99	100
6	7	8	38.46	92.86	2	2
6	7	9	20	83.33	10	15
6	7	10	9.09	73.33	39	31
6	7	11	2.86	63.64	57	42
6	7	12	0	54.55	83	52
6	7	13	0	46.15	98	100
6	8	9	30.3	91.67	1	6
6	8	10	16.22	82.22	20	21
6	8	11	7.5	72.73	32	38
6	8	12	2.38	63.64	65	50
6	8	13	0	55.13	97	66
6	9	10	24.39	91.11	5	10
6	9	11	13.33	81.82	20	30
6	9	12	6.25	72.73	45	43
6	9	13	2	64.1	76	51
6	10	11	20	90.91	10	13
6	10	12	11.11	81.82	20	20

Table 2c. Computational results.

N_1	N_2	N	Common PCs (%)	Available PCs (%)	Error rate, LP approach	Error rate, non-LP approach
6	10	13	5.26	73.08	47	54
6	11	12	16.67	90.91	6	8
6	11	13	9.38	82.05	29	35
6	12	13	14.08	91.03	14	24
7	7	8	55.56	96.43	0	1
7	7	9	31.25	88.89	6	8
7	7	10	16.67	80	16	22
7	7	11	7.69	70.91	45	36
7	7	12	2.44	62.12	68	56
7	7	13	0	53.85	93	66
7	8	9	44.12	94.44	2	3
7	8	10	25.64	86.67	14	20
7	8	11	13.95	78.18	26	23
7	8	12	6.52	69.7	46	40
7	8	13	2.08	61.54	77	55
7	9	10	35.71	93.33	3	7

Table 2c. (continued).

N_1	N_2	N	Common PCs (%)	Available PCs (%)	Error rate, LP approach	Error rate, non-LP approach
7	9	11	21.28	85.45	14	20
7	9	12	11.76	77.27	30	32
7	9	13	5.56	69.23	66	52
7	10	11	29.41	92.73	8	12
7	10	12	17.86	84.85	16	23
7	10	13	10	76.92	41	48
7	11	12	24.59	92.42	7	19
7	11	13	15.15	84.62	20	24
7	12	13	20.83	92.31	9	9
8	8	9	60	97.22	0	0
8	8	10	36.59	91.11	4	6
8	8	11	21.74	83.64	13	22
8	8	12	12	75.76	38	40
8	8	13	5.66	67.95	51	47
8	9	10	48.84	95.56	4	4
8	9	11	30.61	89.09	12	14
8	9	12	18.52	81.82	19	25
8	9	13	10.34	74.36	46	38
8	10	11	40.38	94.55	3	7
8	10	12	25.86	87.88	14	25
8	10	13	15.87	80.77	23	31
8	11	12	33.87	93.94	7	10
8	11	13	22.06	87.18	10	26
8	12	13	28.77	93.59	8	10
9	9	10	63.64	97.78	1	1
9	9	11	41.18	92.73	7	13
9	9	12	26.32	86.36	17	21
9	9	13	16.13	79.49	24	38
9	10	11	52.83	96.36	1	8
9	10	12	35	90.91	14	23
9	10	13	22.73	84.62	17	26
9	11	12	44.44	95.45	4	15
9	11	13	30	89.74	13	22
9	12	13	37.84	94.87	3	6
10	10	11	66.67	98.18	0	0
10	10	12	45.16	93.94	4	10
10	10	13	30.43	88.46	23	26
10	11	12	56.25	96.97	4	3
10	11	13	38.89	92.31	8	16
10	12	13	48	96.15	3	9
11	11	12	69.23	98.48	1	0
11	11	13	48.65	94.87	6	12
11	12	13	59.21	97.44	3	4
12	12	13	71.43	98.72	1	0

consideration. For each such random problem, actual relative weights were assumed as in the numerical example. However, because the Saaty matrices use values from the set $\{1/9, 1/8, \ldots, 1, 2, \ldots, 8, 9\}$, only the random problems which are associated with RCP matrices with entries within the continuous interval $[1/9, 9/1]$ were considered. These computational results are described in more detail in the next section.

5. Computational Results

The computational results are presented in Table 2, which has three parts (Tables 2a, 2b, 2c). The first three columns present the values of N, N_1, N_2, respectively. The fourth column gives the percent of common pairwise comparisons (PCs) of the number of available comparisons. Furthermore, the fifth column presents the percent of available pairwise comparisons of the total number of comparisons. Finally, the last two columns present the number of contradictions (i.e., when the derived

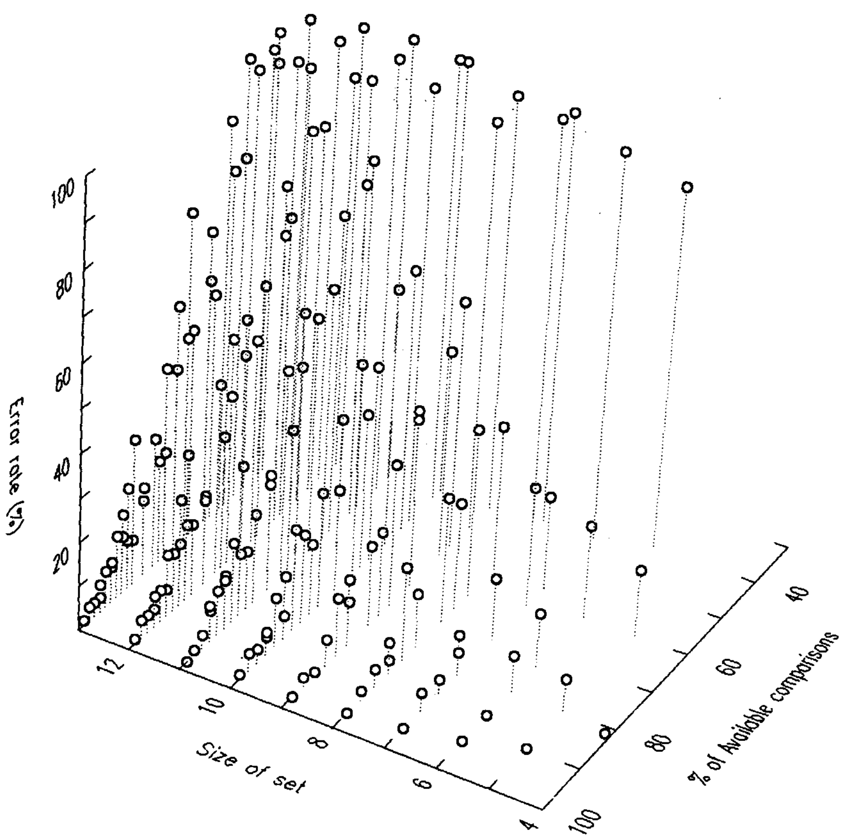

Fig. 2. Error rates under the LP approach for sets of different size versus the available comparisons.

ranking is different than the ranking derived by the original CDP matrix) under the LP and the non-LP approaches.

For instance, consider the seventh row in Table 2a. This row has the numbers [3, 5, 6, 8.33, 80, 8, 10]. In this case, the size of the original set is 6. The first subset includes the first three elements, and the second group includes the last five elements. From the values of N, N_1, N_2, we can verify that the number of available comparisons is 12, the number of common comparisons is 1, and the number of all possible comparisons is 15. Therefore, 8.33% of the available comparisons are common, and the available comparisons represent 80.00% of all possible comparisons. When the LP approach was applied on 100 randomly generated test problems with the previous characteristics, in eight cases the derived ranking was different from the one implied by the corresponding CDP matrices. For the case of using the non-LP approach, however, the same rate was 10%.

These results are also plotted in Figs. 2–7. Figures 2 and 3 depict the error rates, for sets of different size, when the LP and non-LP approaches are used, respectively. In these figures, the error rates are presented as a

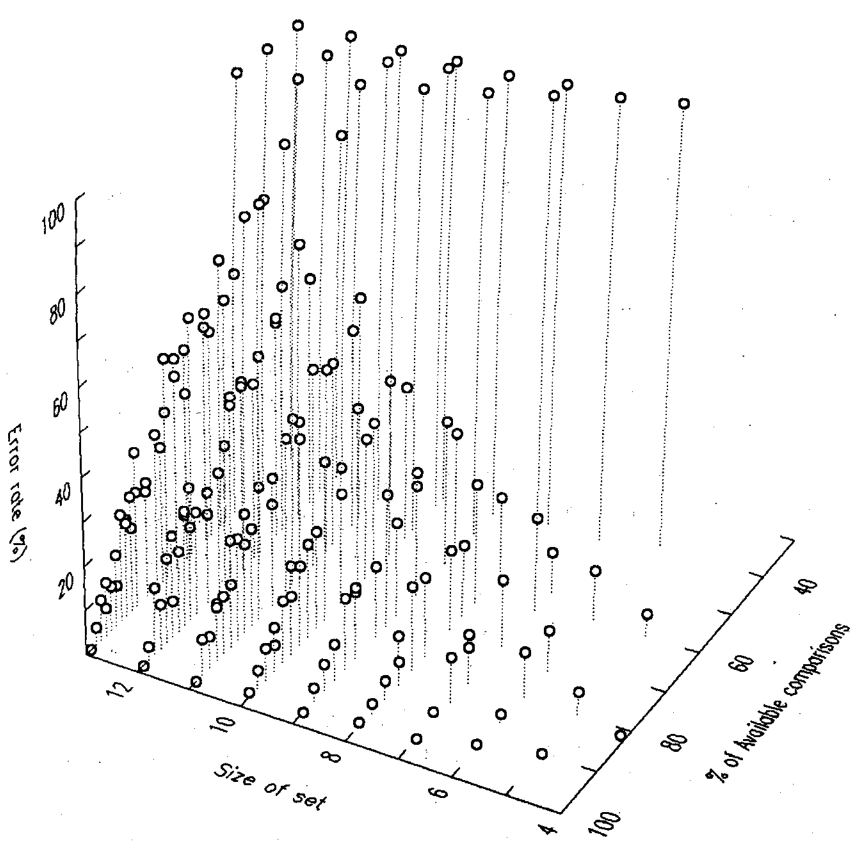

Fig. 3. Error rates under the non-LP approach for sets of different size versus the available comparisons.

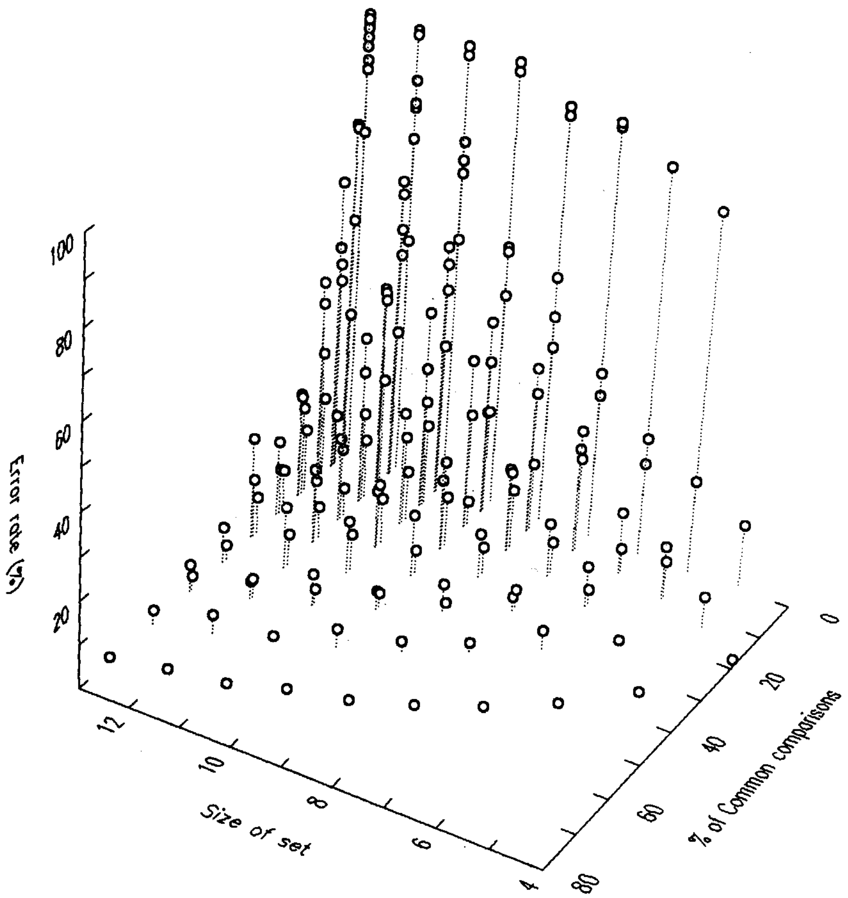

Fig. 4. Error rates under the LP approach for sets of different size versus the common comparisons.

function of the size of the set (i.e., the N value) and the percent of available comparisons (i.e., the fifth column in Table 2). Figures 4 and 5 also depict the error rates for different sets, but now the percent of common comparisons (i.e., the fourth column in Table 2) are used instead. Figures 6 and 7 present the error rates of the two approaches when the results are in terms of the averages of all set sizes. This is the reason why there are only two curves in Figs. 6 and 7; one represents the performance of the LP approach, while the other represents the performance of the non-LP approach. Figure 6 refers to available comparisons, while Fig. 7 refers to common comparisons.

From the previous results, a number of conclusions can be derived. First of all, as is natural, the percent of common comparisons increases with the number of available comparisons. From Figs. 2 to 5, we can observe that, for larger set sizes, the error rates are lower. Furthermore, the LP approach yields, on the average, smaller error rates. The effectiveness of the two approaches becomes more transparent in Figs. 6 and 7. We can

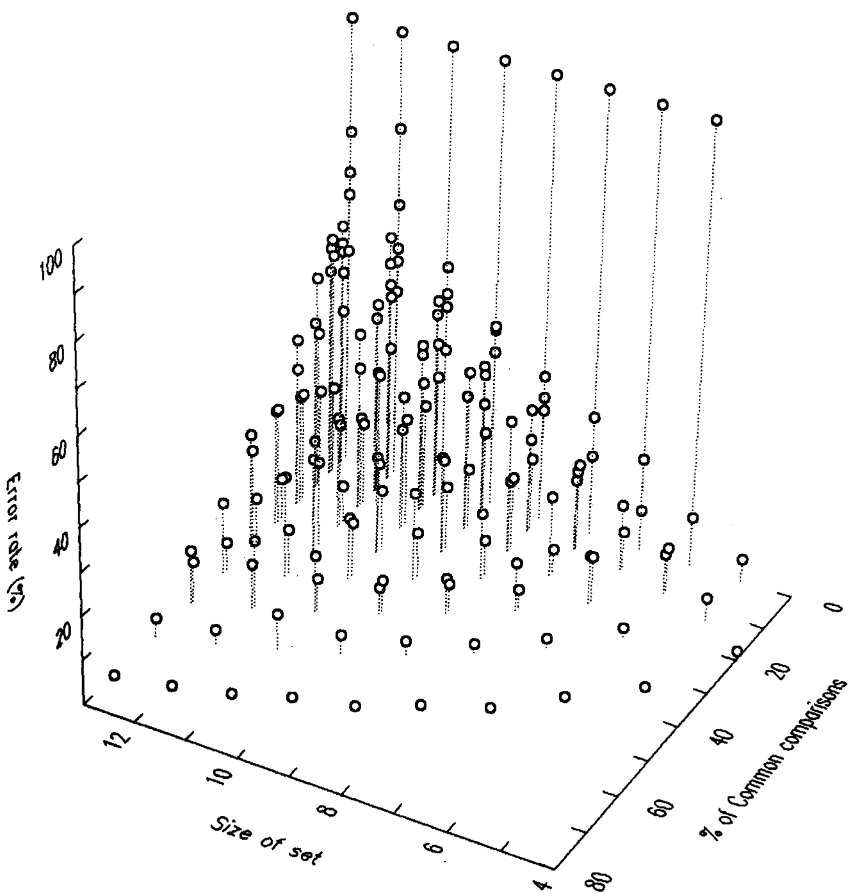

Fig. 5. Error rates under the non-LP approach for sets of different size versus the common comparisons.

observe that, when the percentages of available or common comparisons are low, then either method is unreliable, since it results in too high error rate. For instance, if the percent of available comparisons is less than 50%, then both methods yield error rates higher than 50% (see also Fig. 6).

However, for error rates in lower levels, say less than 30%, the LP approach is always the best method. This is the case where the percent of available comparisons is 80% or higher. This is also the case where we consider the percent of common comparisons. Figure 7 suggests that, when the percent of common comparisons is 10% or higher, then the two methods have error rates less than 20%. Moreover, the LP approach consistently outperforms the non-LP approach. As is anticipated, the performance of the methods converges to being perfect (i.e., the error rates vanish) when the percentages of available or common comparisons approach the upper limit of 100%.

The fact that, in these results, the LP approach performs better than the non-LP approach as the number of common comparisons increases is

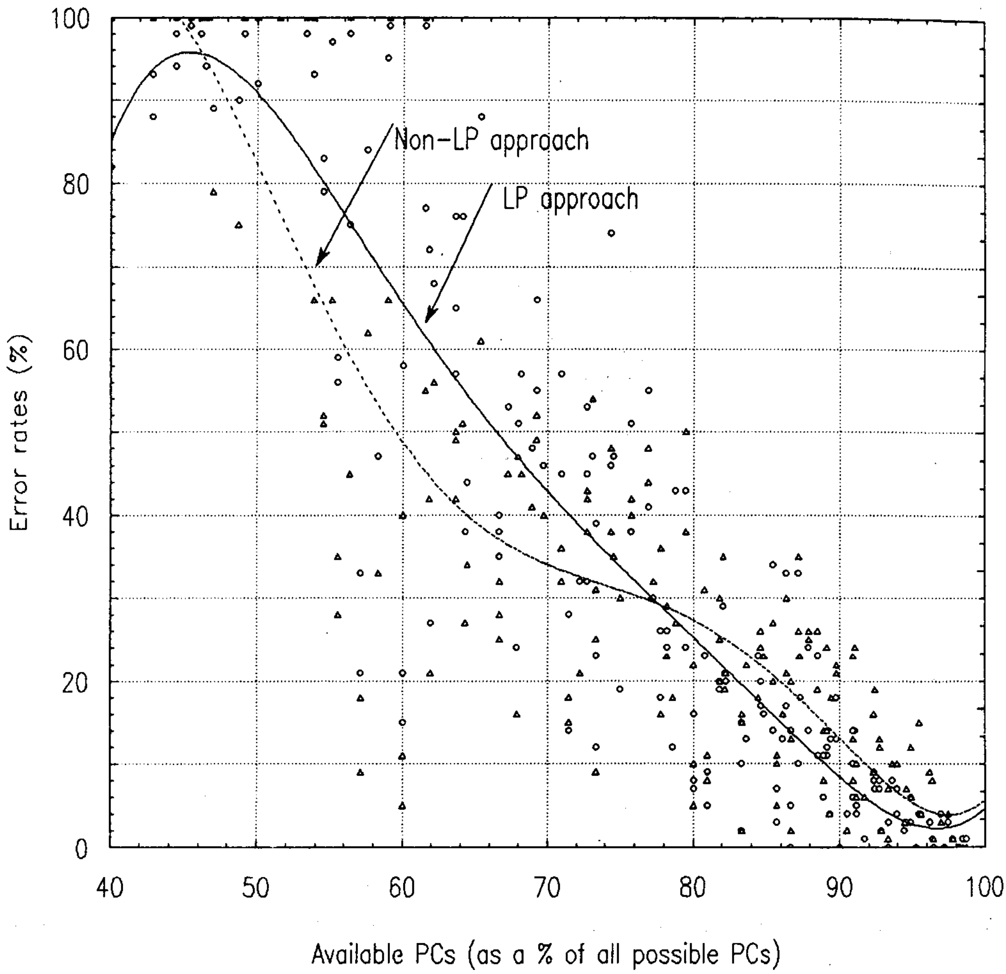

Fig. 6. Error rates for the two approaches versus the available comparisons.

in direct agreement with the way the two approaches were designed. The main difference of the two approaches is that the LP approach utilizes the presence of common comparisons in minimizing the sum of the absolute errors; in the non-LP approach, the issue of the common comparisons is totally ignored. However, it is interesting to observe that, when the percent of common comparisons is very high (more than 60%), then the non-LP approach is as good as the LP approach. Moreover, the non-LP approach is extremely simple [we just need to calculate the means in relations (3)] compared to the more CPU time consuming LP formulation.

6. Concluding Remarks

This paper examined the problem of decomposing a large set of pairwise comparisons in two subsets. These two subsets may have some

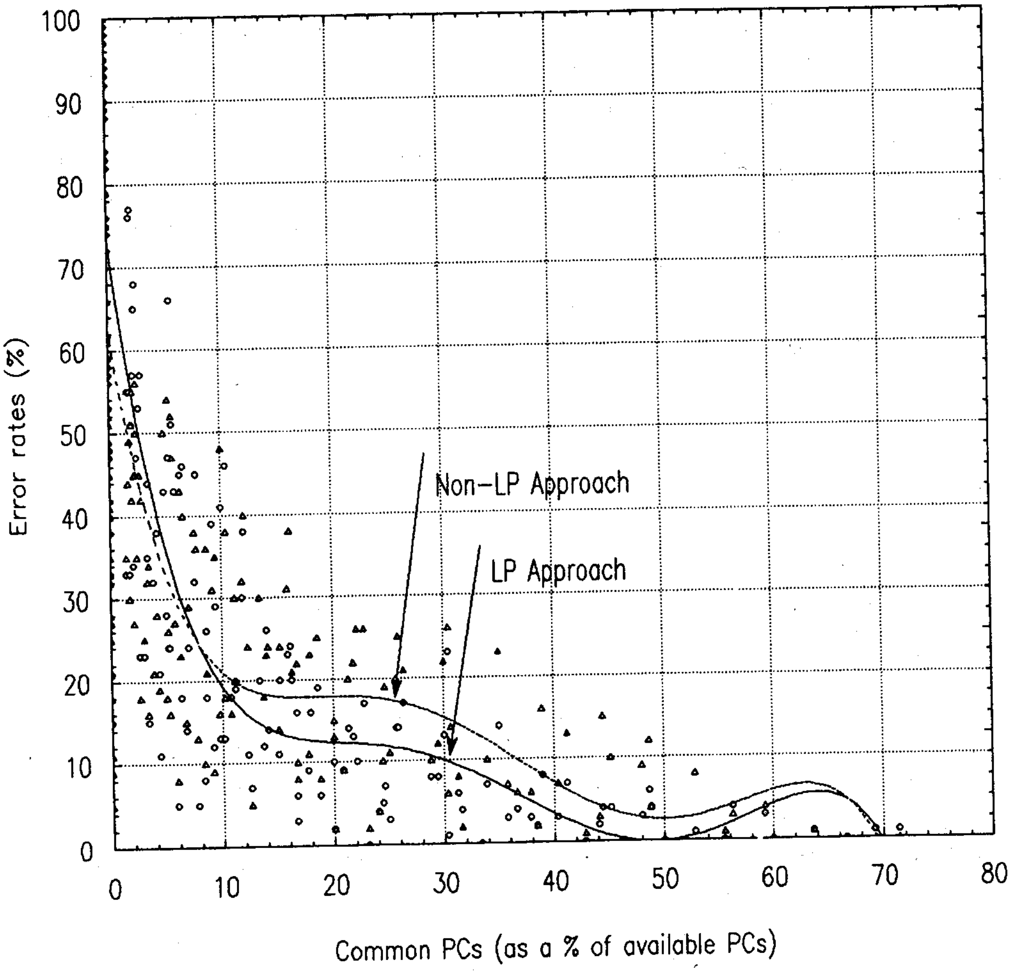

Fig. 7. Error rates for the two approaches versus the common comparisons.

common comparisons. These subsets may be defined by groups of elements which are more similar within these two subsets than when they are considered all together. Two approaches were developed and tested. The first is a simple approach, while the second one is based on an LP formulation. Both approaches first attempt to estimate the missing comparisons and then to derive the relative priorities and rankings by using all the pairwise comparisons.

The LP formulation attempts to minimize the sum of the absolute errors in the missing comparisons. The simulation results suggest that the LP approach is consistently better than the non-LP approach. Furthermore, the effectiveness of this approach improves with the amount of the common comparisons. It should also be stated here that an interesting issue for future research in this area might be how to decompose a large set of elements to be compared into more than two subsets.

Finally, it is important to note that the computational results in this paper are contingent on the way the random data were generated. It is possible that if the random data are generated from other distributions, the results will be different. However, it is anticipated that the LP approach will perform better than the non-LP approach, and moreover its performance to improve with the amount of the input data. Since deriving relative priorities from pairwise comparisons is a popular approach in multicriteria decision making, finding ways of reducing the amount of required data is of critical importance.

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