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Theory and Methodology

On the maximum number of feasible ranking sequences in multi-criteria decision making problems

Evangelos Triantaphyllou [*,1], Bo Shu

Department of Industrial and Manufacturing Systems Engineering, Louisiana State University, College of Engineering, 3128 CEBA Building, Baton Rouge, LA 70803-6409, USA

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Abstract

In many decision problems the focus is on ranking a set of m alternatives in terms of a number, say n, of decision criteria. Given are the performance values of the alternatives for each one of the criteria and the weights of importance of the criteria. This paper demonstrates that if one assumes that the criteria weights are changeable, then the number of all possible rankings may be significantly less than the upper limit of $m!$. As a matter of fact, this paper demonstrates that the number of possible rankings is a function of the number of alternatives and the number of criteria. These findings are important from a sensitivity analysis point of view or when a group decision making environment is considered. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper considers a deterministic decision making problem with m alternatives, denoted as $A_1, A_2, A_3, \ldots, A_m$, to be evaluated in terms of n decision criteria, denoted as $C_1, C_2, C_3, \ldots, C_n$. It is also assumed that the decision maker(s) knows the performance values a_{ij} (for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$) of each one of the alternatives in terms of each one of the decision criteria. Furthermore, we assume that for each decision criterion the decision maker(s) can determine its weight of importance, denoted as w_j (for $j = 1, 2, 3, \ldots, n$). It is further assumed that the weights of importance of the n criteria satisfy the following normalization constraint:

$$\sum_{n}^{j=1} w_j = 1. \tag{1}$$

Often, this kind of problem is difficult from the standpoint that the pertinent data are difficult to be quantified. However, among the previous a_{ij}

* Corresponding author. Tel.: +1-255-388-5372; fax: +1-255-388-5990.
E-mail address: trianta@lsu.edu (E. Triantaphyllou).
[1] URL: http://www.imse.lsu.edu/vangelis.

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and w_j values, one can assert that the most difficult data to be estimated are the weights of the decision criteria (i.e., the w_j values). If one assumes that both the a_{ij} and w_j values are changeable, then it is easy to realize that the number of all possible rankings of the m alternatives is equal to $m!$ (because this is the number of all possible permutations of m objects). However, the number of all possible rankings is different when one considers changes only in the w_j values. This paper investigates this issue and it demonstrates that the number of all possible rankings, when the a_{ij} values are kept constant and the w_j values are allowed to change, may be significantly less than $m!$.

Some popular methods which examine how one can rank alternatives include the weighted sum model (WSM) (Fishburn, 1967) and the analytic hierarchy process (AHP) (Saaty, 1994). The interested reader may want to consult with the surveys reported in Triantaphyllou and Mann (1989) and Chen and Hwang (1992) for additional details. These methods essentially propose to calculate the performance of the alternatives by using an additive function of the following form:

$$P_i = \sum_{j=1}^{n} a_{ij} w_j \quad \text{for } i = 1, 2, 3, \ldots, m, \tag{2}$$

where P_i is the *preference value* of alternative A_i when all the criteria are considered simultaneously. It should also be stated here that we are dealing with cardinal preference values and not preemptive or noncommensurable ones. Then, in the maximization case, the alternatives can be ranked in terms of the previous preference values. Given the ordering

$$P_{i1} \geqslant P_{i2} \geqslant P_{i3} \geqslant \cdots \geqslant P_{im},$$

where $i_1, i_2, i_3, \ldots, i_m$ are the indexes of the m alternatives, the ranking of the m alternatives is

$$A_{i1} \geqslant A_{i2} \geqslant A_{i3} \geqslant \cdots \geqslant A_{im},$$

where " $\geqslant$ " stands for "*better than or equal to*".

In many real world decision making problems, it is even more difficult for the decision maker(s) to determine the criteria weights than the performance values. For example, consider a simplified decision making problem of buying a car. The alternatives (e.g., the different kinds of cars) could be evaluated under two criteria: the gasoline mileage (in miles/gallon) of the cars and the number of years in the initial warranty. Obviously, the performance values of the cars under these two criteria are rather objective and are easy to determine. However, the relative importance of the two criteria can be very subjective and different decision makers may assign totally different values to the criteria weights. Thus, it can be assumed that in this decision matrix the performance values a_{ij} (for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$) are fixed and only the criteria weights w_j (for $j = 1, 2, 3, \ldots, n$) are changeable. From Eq. (1) it can be seen that there are infinitely many different combinations of the w_j values which can satisfy the normalization condition. For each such combination of the w_j values and by using Eq. (2), a different set of performance values P_i can be derived. Next, each such set of P_i values corresponds to a ranking (not necessary unique) of the alternatives.

In the previous example of buying a car, assume that the gasoline mileage and the number of years in the initial warranty for each car are as given in Table 1.

Since there are two decision criteria of heterogeneous units in this problem, first the actual performance values have to be turned into relative values. For instance, if one wishes to use the AHP method, the actual performance values have to be normalized by dividing them by their corresponding sums. For example, the sum of the actual performance values under the first decision criterion is: $20 + 35 + 30 = 85$ (miles/gallon). So the relative performance value of Car A_1 under the first criterion is: $20/85 = 0.24$. Once all the actual performance values have been normalized, the

Table 1
Data for the car selection problem

Alts.	Decision criteria	
	Mileage (miles/gallon)	Number of years in initial warranty
	w_1	w_2
Car A_1	20	5
Car A_2	35	3
Car A_3	30	2

decision matrix of this problem becomes as shown in Table 2.

Furthermore, let us assume that the decision maker regards these two criteria as equally important, so $w_1 = w_2 = 0.5$. Then from Eq. (2), it can be derived that $P_1 = 0.37, P_2 = 0.36$ and $P_3 = 0.28$. Thus, the ranking of the three alternatives is: $A_1 > A_2 > A_3$. However, if the decision maker believes that the first criterion (mileage) is much more important than the second criterion (warranty), he/she may set different values to the criteria weights, say $w_1 = 0.8$ and $w_2 = 0.2$. Accordingly, the preference values will now change as follows: $P_1 = 0.29, P_2 = 0.39$, and $P_3 = 0.32$. That is, now we have $A_2 > A_3 > A_1$, which is different from the first ranking.

Recall that a ranking is a permutation on the order of the alternatives. Therefore, for m alternatives there are $m!$ possible different rankings. As there are theoretically infinitely many possible combinations of the w_j values, it is quite natural to ask if all $m!$ different rankings are feasible when one considers only changes on the criteria weights. Back to the example of buying a car, here m is equal to 3, so there are $3! = 6$ different rankings. Namely the following:

$$A_1 > A_2 > A_3, \quad A_1 > A_3 > A_2, \; A_2 > A_1 > A_3,$$
$$A_2 > A_3 > A_1, \quad A_3 > A_1 > A_2, \text{ and } A_3 > A_2 > A_1.$$

For this particular illustrative example it can be observed that the preference values of alternative A_2 are larger than those of alternative A_3 in terms of both criteria. That is, alternative A_2 dominates alternative A_3. Hence, the two rankings $A_3 > A_2 > A_1$ and $A_3 > A_1 > A_2$ are infeasible in this particular example. That is, not all rankings are feasible.

Table 2
Decision matrix for illustrative example

Alts.	Decision criteria	
	Mileage w_1	Warranty w_2
Car A_1	0.24	0.50
Car A_2	0.41	0.30
Car A_3	0.35	0.20

The case of observing the role of dominated alternatives on the maximum number of rankings may be a trivial one. However, it provides the first motivation that not all $m!$ rankings may be feasible when the w_j values can change while the a_{ij} values are fixed. Similar situations may occur even when no dominated alternatives are present. Thus, the research question examined in this paper is: *when changing the criteria weights arbitrarily, at most how many different rankings of the alternatives can be derived?*

In this paper it is shown that the maximum number of all feasible rankings is a function of both the number of alternatives m and the number of criteria n. Even more surprisingly, this maximum number may be significantly smaller than $m!$. The next sections demonstrate how to calculate this maximum number under different values of m and n.

2. A geometric representation of the maximum number of feasible rankings

2.1. Pairwise comparisons of the alternatives

The ranking of the alternatives can be determined by comparing the preference values of the alternatives two at a time. For example, for three alternatives A_1, A_2 and A_3, one can first compare P_1 with P_2 and then P_1 with P_3. Next, he/she needs to compare P_2 with P_3. Assume that $P_1 > P_2$, $P_1 > P_3$, and $P_2 > P_3$, then $A_1 > A_2 > A_3$. From Eq. (2) we get

$$P_i - P_j = \sum_{k=1}^{n} (a_{ik} - a_{jk}) w_k,$$
$$\text{for } i, j = 1, 2, 3, \ldots, m \text{ and } i \neq j. \qquad (3)$$

If $P_i - P_j > 0$, then A_i is preferred to A_j and vice versa. If $P_i - P_j = 0$, then A_i is as preferable as A_j for the decision maker. Let a_i be the ith row vector of the decision matrix (i.e., $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$). Let W be the weight vector of the decision criteria. That is, $W = (w_1, w_2, \ldots, w_n)$. Then Eq. (3) can be rewritten as follows:

$$P_i - P_j = (a_i - a_j)W^{\mathrm{T}},$$
$$\text{for } i,j = 1,2,3,\ldots,m \text{ and } i \neq j. \quad (4)$$

For $P_i - P_j = 0$, we have

$$(a_i - a_j)W^{\mathrm{T}} = 0. \quad (5)$$

Recall that the a_{ij} values are fixed and the w_j values are changeable, so this problem may be considered geometrically in the space (denoted as E^n) of the n criteria weights. In the E^n space, each configuration of the criteria weights $(w_1, w_2, \ldots, w_n)$ represents a distinct point. For the convenience of discussion, the set of points which satisfy the normalization condition will be called the *base-plane* of the decision problem:

Definition 1. The $(n-1)$-dimension hyperplane which is defined by Eq. (1) and the nonnegativity constraints $w_j \geqslant 0$ is called the base-plane of a problem with n decision criteria.

Essentially, the base-plane is an $(n-1)$-dimension hyperplane with the property that each point on that plane satisfies Eq. (1). If there are only two decision criteria, then this is a line segment (segment AB in Fig. 1). If there are three decision criteria, then this is the area of a 2-dimension (2-D) plane (plane area ABC in Fig. 2).

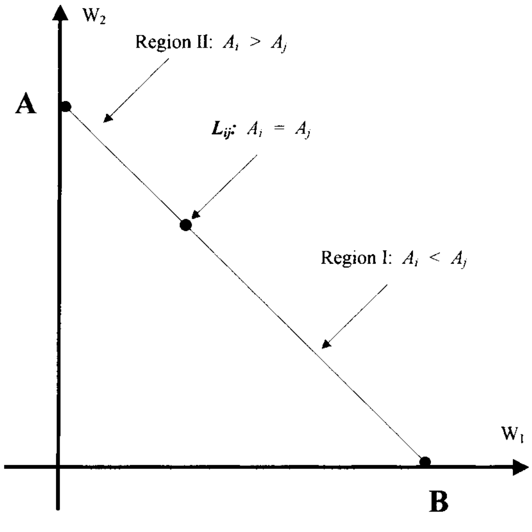

Fig. 1. Bipartition of a base-plane in the 2-D space.

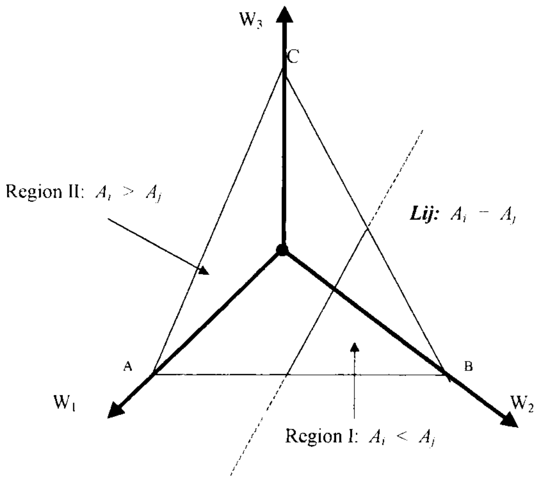

Fig. 2. Bipartition of a base-plane in the 3-D space.

Similarly, the $(n-1)$-dimension hyperplane of the pairwise comparison between alternatives A_i and A_j is defined by Eq. (5). On this plane A_i is equal to A_j in preference value. As the condition of normalization has to be satisfied, one can get from Eqs. (5) and (1), and the nonnegativity constraints

$$\left\{ \begin{array}{r} (a_i - a_j)W^{\mathrm{T}} = 0 \\ w_1 + w_2 + w_3 + \cdots + w_n = 1 \\ w_1, w_2, w_3, \ldots, w_n \geqslant 0 \end{array} \right\}, \quad (6)$$

where $i,j = 1,2,3,\ldots,m$ and $i \neq j$.

From the geometric point of view the set of solutions to Eq. (6) is an $(n-2)$-dimension hyperplane which is the intersection of the base-plane and the hyperplane of the pairwise comparison. If this set of solutions is nonempty, we call it the *separating plane between alternatives A_i and A_j*, and it will be denoted as L_{ij}. Therefore, the L_{ij} plane will divide the base-plane into distinct regions. In one region (say region I) A_j is always better than A_i while in the opposite region (say region II) A_j is always worse than A_i and on the boundary of these two regions (i.e., on L_{ij}) A_i is equal to A_j in preference value. The geometrical representation of the pairwise comparison between two alternatives A_i and A_j is depicted in Figs. 1 and 2, where decision problems with two and three criteria are considered, respectively.

If there is no feasible solution to Eq. (6), then there will not be any intersection of the pairwise comparison plane and the base-plane. As a result, L_{ij} will not exist. In this situation the base-plane is on one side of the pairwise comparison plane and thus either A_i is always better than A_j or A_j is always better than A_i.

2.2. Geometric representation of rankings

To determine the complete ranking of the m alternatives, the final preference values of the alternatives have to be compared in a pairwise manner by performing the following set of pairwise comparisons: (P_i with P_j) for $i = 1, 2, 3, \ldots, m-1$, $j = 1, 2, 3, \ldots, m$, and $i < j$.

There are $k = m(m-1)/2$ such pairwise comparisons. Since each pairwise comparison corresponds to a separating plane, there are at most k separating planes denoted as L_{ij} (for $i, j = 1, 2, \ldots, m$ and $i < j$) as defined by Eq. (6). These separating planes intersect with the base-plane. Thus, the separating planes will divide the base-plane into several sub-regions. The following three lemmas follow easily from the previous definitions and thus are stated without the proofs.

Lemma 1. *Any point on the base-plane corresponds to a ranking of the alternatives.*

Lemma 2. *The points in the same sub-region of the base-plane correspond to the same ranking of the alternatives, and vice versa.*

Lemma 3. *Points in different sub-regions correspond to different rankings of the alternatives.*

From these lemmas it follows that there is an one-to-one mapping from the sub-regions on the base-plane to the rankings of the alternatives. We call these sub-regions the *ranking regions* of the decision problem. Therefore, the number of all possible rankings of the m alternatives is equal to the number of the ranking regions on the base-plane. Hence, the problem can be changed into the geometrical problem of finding how many ranking regions a base-plane can be divided into by the separating planes.

2.3. Partition of the weight space

From the previous discussion it follows that geometrically speaking the partition of the base-plane by the k separating planes is the same problem as the partition of an $(n-1)$-dimension hyperplane by k hyperplanes of $(n-2)$-dimension each. Note that for a specific decision matrix, the number of separating planes may be less than $k(= m(m-1)/2)$. In this paper the objective is to investigate, for a given decision matrix, the maximum number of rankings (which also corresponds to the maximum number of ranking regions on the base-plane). Therefore, the goal is to try to divide the base-plane into as many ranking regions as possible. The more separating planes there are, the more regions these planes can divide the base-plane into. Hence, only those decision matrices with all the $m(m-1)/2$ separating planes have the opportunity to yield the maximum number of rankings when the criteria weights can change.

The number of ranking regions also depends on the relative positions of these separating planes on the base-plane. For example, two parallel lines can only divide a plane into three regions while two intersecting lines can divide a plane into four regions. From a pure geometrical point of view, the more intersections these separating planes have with each other, the more ranking regions they can divide the base-plane into. In Brualdi (1979) the concept of *general position* is defined to describe the positions among k hyperplanes of dimension n each.

Definition 2. A set of k hyperplanes of dimension n is said to be in the general position, if every two of them meet at an $(n-1)$-dimension hyperplane but no three of them meet at such an $(n-1)$-dimension hyperplane.

For illustrative purposes one may consider the lines in Figs. 3 and 4. These lines are 1-D planes and they can represent the separating planes of a

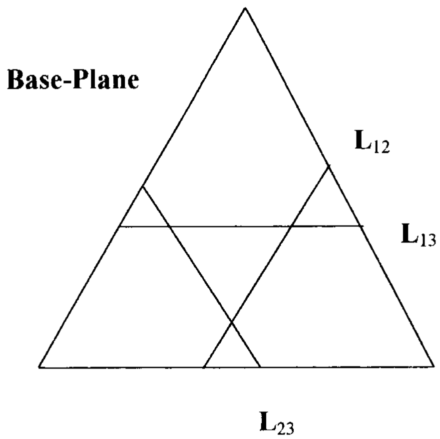

Fig. 3. Lines in general position.

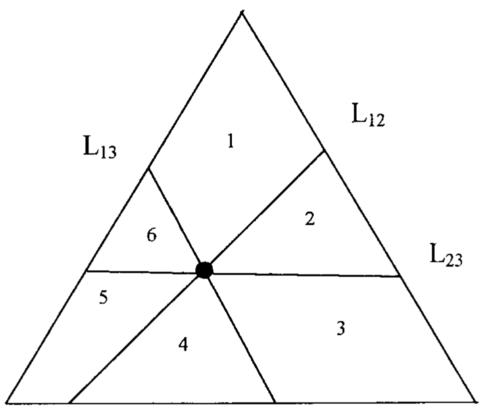

Fig. 4. Lines not in general position.

decision problem with three alternatives. Each figure also depicts the base-plane. The lines (separating planes) in Fig. 3 are in the general position, while the ones in Fig. 4 are not in the general position. Furthermore, it can be shown that the situation depicted in Fig. 3 is infeasible in decision problems. This is formally stated in Theorem 1.

Theorem 1. *The separating planes of a multi-criteria decision making problem can never be in the general position.*

Proof. We will use the approach of logical contradiction. Suppose that in some problem the separating planes are in the general position. Consider a set of three alternatives. Without loss of generality, we call these alternatives A_1, A_2, and A_3. By Definition 2, the separating planes (denoted as L_{12}, L_{13}, and L_{23}) for these alternatives intersect with each other but do not cross through the same point. Consider the point of intersection of any two of them, say separating planes L_{12} and L_{13}. By using the definition of a separating plane, it can be easily derived that at that point the following is true: $A_1 = A_2$ and $A_1 = A_3$. Therefore, we should also have $A_2 = A_3$. Thus, the third separating plane, L_{23}, should be crossing through the same intersection point. However, this is in direct contradiction with the initial assumption that all the separating planes are in the general position. Thus, the separating planes can never be in the general position and thus Theorem 1 has been proved. □

The base-plane in Fig. 4 is divided into six ranking regions each of which corresponds to one of the six possible rankings of the three alternatives. Namely:

$$A_1 > A_2 > A_3, \quad A_1 > A_3 > A_2, \quad A_2 > A_1 > A_3,$$
$$A_2 > A_3 > A_1, \quad A_3 > A_2 > A_1, \quad A_3 > A_1 > A_2.$$

2.4. A theoretical analysis of the number of feasible rankings

The number of feasible rankings for an MCDM problem has an interesting property: if the number of decision criteria (n) is greater than or equal to the number of alternatives (m), then we can always find a decision matrix (not necessarily unique) such that by changing the weights of the decision criteria arbitrarily while keeping the performance values fixed, we can get all the $m!$ rankings of the alternatives. However, if $m > n$ then it is impossible to get all the $m!$ rankings by simply changing the weights of the decision criteria. This property is formally stated in Theorems 2 and 3. Before these theorems are stated, we introduce some relevant lemmas:

Lemma 4. *Let $A_{i1} \geqslant A_{i2} \geqslant \cdots \geqslant A_{ij} \geqslant A_{i(j+1)} \geqslant \cdots \geqslant A_{im}$ and $A_{i1} \geqslant A_{i2} \geqslant \cdots A_{i(j+1)} \geqslant A_{ij} \geqslant \cdots \geqslant A_{im}$ be two feasible rankings which differ by only one pair of alternatives (i.e., A_{ij} and $A_{i(j+1)}$). Then there exists at least one point on the base-plane such that $A_{i1} \geqslant A_{i2} \geqslant \cdots \geqslant A_{ij} = A_{i(j+1)} \geqslant \cdots \geqslant A_{im}$ (for $i_1, i_2, \ldots, i_m = 1, 2, 3, \ldots, m$).*

Proof. Without loss of generality, let $A_1 \geqslant A_2 \geqslant \cdots \geqslant A_j \geqslant A_{(j+1)} \geqslant \cdots \geqslant A_m$ and $A_1 \geqslant A_2 \geqslant \cdots \geqslant A_{(j+1)} \geqslant A_j \geqslant \cdots \geqslant A_m$ be the two feasible rankings on the base-plane. These two rankings correspond to two ranking regions separated by separating plane $L_{j(j+1)}$. Since the points along this plane correspond to $A_j = A_{(j+1)}$, it is concluded that the point(s) which lay on the separating plane $L_{j(j+1)}$ and are adjacent to these two ranking regions possess the desirable property. □

Lemma 5. *If all the m! rankings are feasible, then there must exist at least one point on the base-plane such that $A_1 = A_2 = A_3 = \cdots = A_m$.*

Proof. Since all the rankings are feasible, the following two rankings $R_1 = \{A_1 \geqslant A_2 \geqslant A_3 \geqslant \cdots \geqslant A_m\}$ and $R_2 = \{A_2 \geqslant A_1 \geqslant A_3 \geqslant \cdots \geqslant A_m\}$ are also feasible, since R_1 and R_2 differ by only one pair of alternatives (i.e., A_1 and A_2). Thus, from Lemma 4, the ranking $R_{1.2} = \{A_1 = A_2 \geqslant A_3 \geqslant \cdots \geqslant A_m\}$ is feasible as well. Also, the rankings $R'_1 = \{A_3 \geqslant A_1 \geqslant A_2 \geqslant \cdots \geqslant A_m\}$ and $R'_2 = \{A_3 \geqslant A_2 \geqslant A_1 \geqslant \cdots \geqslant A_m\}$ are feasible. Similarly, from Lemma 4, the ranking $R'_{1.2} = \{A_3 = A_1 \geqslant A_2 \geqslant \cdots \geqslant A_m\}$ is also feasible. As both rankings $R_{1.2}$ and $R'_{1.2}$ are feasible, then from Lemma 4 it can be concluded that the ranking $R_3 = \{A_1 = A_2 = A_3 \geqslant \cdots \geqslant A_m\}$ is also feasible. Thus, working as in the above procedure, finally we can get that the following ranking: $R = \{A_1 = A_2 = A_3 = \cdots = A_m\}$ is also feasible. Hence, Lemma 5 has been proved. □

Lemma 6. *Let α_i $(i = 1, 2, \ldots, m)$ be the i-th row of a decision matrix. If $(\alpha_i - \alpha_j)$ can be represented as a linear combination of some vectors $(\alpha_i - \alpha_{k1}), (\alpha_i - \alpha_{k2}), \ldots, (\alpha_i - \alpha_{kp})$, where $k_1, k_2, \ldots, k_p$ are indices of alternatives and $k_1, k_2, \ldots, k_p \neq i, j$, then some of the rankings will be infeasible.*

Proof. Assume that all the rankings are feasible. Then this will lead into a logical contradiction. Since $(\alpha_i - \alpha_j)$ can be represented as a linear combination of $(\alpha_i - \alpha_{k1}), (\alpha_i - \alpha_{k2}), \ldots, (\alpha_i - \alpha_{kp})$, we get

$$(\alpha_i - \alpha_j) = c_1(\alpha_i - \alpha_{k1}) + c_2(\alpha_i - \alpha_{k2}) + \cdots + c_p(\alpha_i - \alpha_{kp}), \tag{i}$$

where $k_1, k_2, \ldots, k_p$ are indices between 1 and m and $k_1, k_2, \ldots, k_p \neq i, j$ and $c_1, c_2, \ldots, c_p$ are non-zero coefficients. From Eq. (4) we get: $P_i - P_j = (\alpha_i - \alpha_j)W^T$. By post multiplying both sides of (i) by W^T, we get

$$P_i - P_j = c_1(P_i - P_{kl}) + c_2(P_i - P_{k2}) + \cdots + c_p(P_i - P_{kp}). \tag{ii}$$

Since it was assumed that all the rankings are feasible, then we can always find a weight vector W^* such that:

$P_i^* < P_j^*$ and

$P_i^* > P_{kl}^*$ if $c_1 > 0$,

$P_i^* < P_{kl}^*$ if $c_1 < 0$,

where $l = 1, 2, 3, \ldots, p$ and P_i^* is the preference value of alternative A_i under the criteria weights in vector W^*.

Therefore, when the weight vector W^* is used, then the right-hand side of Eq. (ii) will be positive while its left-hand side will be negative. This is a logical contradiction. Hence, the assumption that all the rankings are feasible is not valid and Lemma 6 has been proved. □

Lemma 6 is further illustrated with the example given in Table 3 which has 3 alternatives and 3 criteria.

From Table 3 it follows that: $\alpha_1 = (3, 4, 5)$, $\alpha_2 = (6, 1, 2)$ and $\alpha_3 = (4, 3, 4)$. Thus we have $(\alpha_1 - \alpha_2) = (-3, 3, 3)$ and $(\alpha_1 - \alpha_3) = -(-1, 1, 1)$. Therefore, $(\alpha_1 - \alpha_2) = 3(\alpha_1 - \alpha_3)$. Since $P_1 = 3w_1 +$

Table 3
An example of some dependent preference values

Alts.	Decision criteria		
	C_1	C_2	C_3
	w_1	w_2	w_3
A_1	3	4	5
A_2	6	1	2
A_3	4	3	4

$4w_2 + 5w_1$, $P_2 = 6w_1 + w_2 + 2w_3$ and $P_3 = 4w_1 + 3w_2 + 4w_3$, it follows that: $P_1 - P_2 = 3(P_1 - P_3)$. From Lemma 6, it can be seen that the ranking $A_2 > A_1 > A_3$ is infeasible because whenever $A_1 > A_3$, then we also have $A_1 > A_2$.

Theorem 2. *Let $F_n(m)$ be the maximum number of the ranking regions on the base-plane. If $m > n$, then $F_n(m) < m!$.*

Proof. Consider the following system of linear equations and the nonnegativity constraints

$$\left\{\begin{array}{r}(a_1 - a_2)W^{\mathrm{T}} = 0\\(a_1 - a_3)W^{\mathrm{T}} = 0\\\vdots\\(a_1 - a_m)W^{\mathrm{T}} = 0\\w_1 + w_2 + w_3 + \cdots + w_n = 1\\w_1, w_2, w_3, \ldots, w_n \geqslant 0\end{array}\right\}, \tag{7}$$

where α is the ith row of the decision matrix and $W = (w_1, w_2, w_3, \ldots, w_n)$ is the vector with the criteria weights. Equation $(\alpha_1 - \alpha_i)W^{\mathrm{T}} = 0$ defines the separating plane L_{1i} (for $i = 2, 3, \ldots, m$) of alternatives A_1 and A_i. Since $(\alpha_1 - \alpha_2), \ldots, (\alpha_1 - \alpha_m)$ are row vectors of dimension n and $m > n$, there are at most n of them that can be linearly independent (Roman, 1987). Since $m > n$, then $m - 1 \geqslant n$. If among these $(m - 1)$ vectors, we can find exactly n vectors that are linearly independent, then there will be no solution to system (7). Geometrically, this means that there is no point on the base-plane such that all the alternatives are equal to each other. From Lemma 5, it can be seen that in this situation, one can never get all the $m!$ possible rankings of the m alternatives. Thus, $F_n(m) < m!$.

If there are less than n vectors that are linearly independent, then system (7) can have feasible solutions. Without loss of generality, we assume that $(\alpha_1 - \alpha_2), (\alpha_1 - \alpha_3), \ldots, (\alpha_1 - \alpha_{k+1})$ (where $k < n$) are linearly independent. Therefore, each one of the row vectors $(\alpha_1 - \alpha_{k+2}), (\alpha_1 - \alpha_{k+3}), \ldots, (\alpha_1 - \alpha_m)$ can be represented as a linear combination of the vectors $(\alpha_1 - \alpha_2)$, $(\alpha_1 - \alpha_3), \ldots, (\alpha_1 - \alpha_{k+1})$. By Lemma 6, one can conclude that some rankings will be infeasible and hence $F_n(m)$ will be less than $m!$. Thus Theorem 2 has been proved. □

If $m \leqslant n$, then system (7) always has a solution but it is not necessarily feasible for a particular decision matrix (i.e., the nonnegativity constraints $w_i \geqslant 0$ may not be satisfied). However, for any given m and n values, one can always find such a decision matrix for which system (7) will have feasible solutions. This is formally stated in Lemma 7.

Lemma 7. *If $m \leqslant n$, then there exists a decision matrix A such that*: (i) rank$(A) = m$ *and* (ii) *system* (7) *has a feasible solution.*

Proof. Let $W^* = (w_1^*, w_2^*, \ldots, w_n^*)$ be a criteria weight vector such that $w_1^* + w_2^* + \cdots + w_n^* = 1$ and $w_i^* \geqslant 0$. Lemma 7 can be proved by constructing a decision matrix A such that: (i) its rank is m; (ii) the solution to system (7) is W^*.

Next we consider a vector β in E^n such that

$$W^*\beta^{\mathrm{T}} = 0, \tag{8}$$

Here W^* is known and β is unknown. Clearly, there are infinitely many vectors β which can satisfy Eq. (8). However, since rank $(W^*) = 1$, among these vectors, there are $(n - 1)$ and only $(n - 1)$ of them that can be linearly independent with each other (Roman, 1987). As $m \leqslant n$, one can always pick up $(m - 1)$ such vectors. Let $\beta_2, \beta_3, \ldots, \beta_m$, be these vectors so that $W^*\beta_i^{\mathrm{T}} = 0$ (for $i = 2, 3, \ldots, m$) and they are linearly independent with each other. Recall that there are at most n independent n-dimension vectors in the E^n space. Since $m \leqslant n$ and β_i (for $i = 2, 3, \ldots, m$) are vectors in the E^n space, we can always find another n-dimension row vector α_1' such that $\alpha_1', \beta_2, \beta_3, \ldots, \beta_m$ are also linearly independent with each other.

Next, we construct the decision matrix A by using vectors $\alpha'_1, \beta_2, \beta_3, \ldots, \beta_m$. Let α_i ($i = 1, 2, \ldots, m$) be the ith row of decision matrix A. Then $\alpha_1 = \alpha'_1$, $\alpha_2 = \alpha'_1 - \beta_2, \ldots, \alpha_i = \alpha'_1 - \beta_i, \ldots,$ and $\alpha_m = \alpha'_1, -\beta_m$. Since $\alpha'_1, \beta_2, \beta_3, \ldots,$ and β_m are linearly independent, $\alpha_1, \alpha_2, \ldots, \alpha_m$, are also linearly independent. Therefore, $\text{rank}\,(A) = m$ and condition (iv) is satisfied. Also, $(\alpha_1 - \alpha_i) = \beta_i$ (for $i = 2, 3, \ldots, m$) and $W^* \beta_i^T = 0$, hence the newly constructed decision matrix W^* is a feasible solution to system (7) and condition (8) is satisfied. Thus, Lemma 7 has been proved. □

Now we consider an example with $m = 3$ and $n = 4$. One can arbitrarily select a weight vector, say $W^* = (0.25, 0.25, 0.25, 0.25)$. Next we construct a decision matrix A such that A can yield a feasible solution to system (7) equal to W^*. Since $W^* \beta_i^T = 0$ (for $i = 2, 3, \ldots, m$), we have

$$0.25\beta_{i1} + 0.25\beta_{i2} + 0.25\beta_{i3} + 0.25\beta_{i4} = 0. \qquad (9)$$

Among the β_i vectors which satisfy Eq. (9), we select $(m - 1 = 2)$ of them which are linearly independent. Let β_2 and β_3 be such two vectors. For instance, let: $\beta_2 = (1, 1, 1, -3)$, and $\beta_3 = (1, -3, 1, 1)$. Next, we find another vector α'_1 that is linearly independent with β_2 and β_3. Let $\alpha'_1 = (2, 2, 2, 2)$. Finally, we construct the decision matrix A such that: $\alpha_1 = \alpha'_1 = (2, 2, 2, 2)$, $\alpha_2 = \alpha'_1 - \beta_2 = (1, 1, 1, 5)$ and $\alpha_3 = \alpha'_1 - \alpha_3 = (1, 5, 1, 1)$. Hence, the decision matrix is as in Table 4.

It is easy to verify that this matrix is of rank 3 and the solution of system (7) is $W^* = (0.25, 0.25, 0.25, 0.25)$. Next, we prove that for such a decision matrix all the $m!$ rankings are feasible.

Table 4
Sample decision matrix satisfying the conditions of Lemma 7

Alts.	Decision criteria			
	C_1	C_2	C_3	C_4
	w_1	w_2	w_3	w_4
A_1	2	2	2	2
A_2	1	1	1	5
A_3	1	5	1	1

Table 5
Different criteria weights and corresponding rankings

Criteria weights (w_1, w_2, w_3, w_4)	Rankings
(0.5, 0.3, 0.1, 0.1)	$A_3 > A_1 > A_2$
(0.5, 0.2, 0.1, 0.1)	$A_1 > A_3 > A_2$
(0.2, 0.4, 0.1, 0.3)	$A_3 > A_2 > A_1$
(0.4, 0.1, 0.3, 0.2)	$A_1 > A_2 > A_3$
(0.2, 0.2, 0.3, 0.3)	$A_2 > A_1 > A_3$
(0.2, 0.2, 0.1, 0.4)	$A_2 > A_3 > A_1$

Theorem 3. *If $m \leqslant n$, then one can always find a decision matrix for which all the $m!$ rankings are feasible and $F_n(m) = m!$.*

Proof. Consider a decision matrix which satisfies the following two conditions: (i) its row vectors α_i (for $i = 1, 2, \ldots, m$) are linearly independent, and (ii) system (7) has feasible solutions. Recall that from Lemma 7 it is always possible to find such a decision matrix. By Lemma 5, the second condition is necessary for the decision matrix to have $m!$ feasible rankings. The first condition guarantees that all the separating planes are linearly independent and hence the difference between the preference values of alternatives A_i and A_j (for $j = 2, 3, 4, \ldots, m$ and $j \neq i$) are also independent of each other. From Lemma 6 it can be seen that in this situation all the $m!$ rankings are feasible for this decision matrix. Thus Theorem 3 has been proved. □

The decision matrix in Table 4 has three independent rows and it can yield a feasible solution ($W^* = (0.25, 0.25, 0.25, 0.25)$) to system (7). By Theorem 3, all the $3! = 6$ rankings are feasible for this decision matrix. In Table 5, the six possible rankings of the decision making problem depicted in Table 4 are listed when different values are assigned to the criteria weights.

3. A decomposition method for calculating $F_n(m)$

3.1. Partition of a space by hyperplanes in the general position

As it has been proved in Theorem 2, if $n < m$, then $F_n(m)$ will be less than $m!$. Since the maximum number of feasible rankings is equal to the maximum number of ranking regions that the separating planes can divide a base-plane into, the value of $F_n(m)$ can be calculated by properly utilizing its geometrical interpretation.

First, let us consider the special case of the space partition problem, where all the separating planes *are assumed to be in the general position.* In this special case, the maximum number of sub-spaces that the separating planes divide the base-plane into can be calculated analytically. We first need to study this special case in order to develop some formulas that will be used next for the actual case which requires that the separating planes *not to be* in the general position (Theorem 1). The following discussion is a brief introduction of Brualdi's (1979) solution to this problem.

Let q (where $q = 1, 2, 3, \ldots$) be the number of $(i-1)$-dimension separating hyperplanes all of which are in the general position. Let $h_i(q)$ (for $i = 1, 2, 3, \ldots, m$) be the maximum number of sub-spaces that these separating planes can divide the i-dimension space into.

For $i = 1$, $h_1(q)$ is simply the number of segments into which a line is divided by q distinct points. Thus, we have $h_i(0) = 1$ and $h_1(1) = 2$. Let $q \geqslant 1$, and consider a line divided into $h_1(q-1)$ segments by $q-1$ points. If the qth point is inserted on the line, then one of the existing segments is divided into two segments. Hence

$$h_1(q) = h_1(q-1) + 1. \tag{10}$$

For $i = 2$, a plane is divided into $h_2(q-1)$ regions by $q-1$ lines in the general position. Now we insert the qth line in the plane so that the q lines are in the general position. The first $q-1$ lines intersect with the qth line in $q-1$ distinct points which divide the line into $h_1(q-1)$ segments. Each one of these segments of the qth line divides an existing region of the plane into two regions. Hence

$$h_2(q) = h_2(q-1) + h_1(q-1). \tag{11}$$

The results can be generalized to higher dimensions. Thus, for $i \geqslant 1$ we have

$$h_i(q) = h_i(q-1) + h_{i-1}(q-1). \tag{12}$$

In Brualdi (1979) there is a detailed description of how to solve this recurrent function. In this paper we just omit the procedure for solving this recurrent function and provide the final solution

$$h_i(q) = \binom{q}{0} + \binom{q}{1} + \cdots + \binom{q}{i}. \tag{13}$$

Therefore, if the separating planes are in the general position, then for the $(n-1)$-dimension space with k $(n-2)$-dimension separating planes the maximum number of sub-spaces can be calculated as follows:

$$h_{n-1}(k) = \binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{n-1}. \tag{14}$$

3.2. Partition of the base-plane by separating planes when $n \leqslant 3$

As it has been discussed in Section 2.3, to divide the base-plane into more sub-regions, there should be more intersections among these separating planes. It is obvious that if these separating planes could be in the general position, then they would have the maximum number of distinctive intersections and hence could divide the base-plane into the maximum number of sub-regions. For the decision making model assumed in this paper, as there are n decision criteria, the base-plane is of $(n-1)$ dimension and the separating planes are of $(n-2)$ dimension. In this situation, Eq. (14) can be applied directly and the problem is trivial. However, by Theorem 1, the separating planes of an MCDM problem can never be in the general position. Therefore, Eq. (14) *cannot be applied* directly and some other method has to be found to calculate $F_n(m)$.

For $n = 1$, clearly there will be only one ranking sequence available. For $n = 2$, the value of $F_n(m)$ can be calculated directly from its geometric meaning as depicted in Fig. 5. The separating planes $L_{12}, L_{13}, \ldots, L_{(m-1)m}$ divide the base-plane: $w_1 + w_2 = 1.00$ into $(k+1)$ segments, where $k = m(m-1)/2$. Hence

$$F_2(m) = \binom{m}{2} + 1. \tag{15}$$

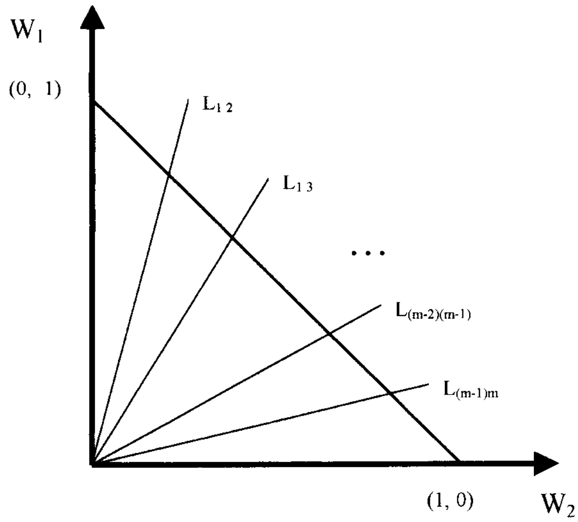

Fig. 5. The case of m alternatives with 2 criteria.

For $n = 3$, $F_n(m)$ is equal to the maximum number of ranking regions that k lines can divide a 2-D plane into. Since we cannot use Eq. (14) directly, a decomposition method is proposed to calculate the value of $F_n(m)$. Thus, instead of placing all the separating planes onto the base-plane and investigating how many ranking regions they can divide the base-plane into, we will decompose the dividing process of the base-plane by these separating planes into k (where $k = m(m-1)/2$) steps and at each step, only one separating plane will be added onto the base-plane. Also when a new separating plane is added onto the base-plane, it should intersect with all the existing planes on the base-plane and have as many distinct intersections with the existing planes as possible. In this way, when all the separating planes are added onto the base-plane, the base-plane is divided into the maximum number of ranking regions.

Each time a new separating plane is added we calculate how many more ranking regions can this plane divide the base-plane into. Finally, when all the separating planes are added onto the base-plane, the number of ranking regions that the base-plane is divided into is the sum of the number of new regions that are obtained at each step. It is obvious that the value of $F_n(m)$ obtained from this process should be the same as if one counts the number of ranking regions when all the k planes are already located on the base-plane.

The k separating planes will be placed onto the base-plane in the sequence: $(L_{12}, L_{13}, L_{14}, \ldots, L_{1m})$, $(L_{23}, L_{24}, \ldots, L_{2m})$, $(L_{34}, L_{35}, \ldots, L_{3m})$, $\ldots$, and $L_{(m-1)m}$. Let K_{ij} be the number of separating planes that are already on the base-plane before L_{ij} is added. Then, when all the separating planes are added onto the base-plane, the maximum number $F_3(m)$ of ranking regions can be calculated as follows:

$$F_3(m) = h_2(m-1) + \sum_{i=2}^{m-1} \sum_{j=i+1}^{m} h_1(K_{ij} - i + 1). \quad (16)$$

Eq. (16) can be further reduced to the following equation:

$$F_3(m) = \binom{k}{0} + \binom{k}{1} + \binom{k}{2} - \binom{m}{3}, \quad (17)$$

where $k = m(m-1)/2$. The detailed proofs of Eqs. (16) and (17) are omitted in this paper but are available from the first author.

3.3. An example with 4 alternatives and 3 criteria

Now we consider an example with 4 alternatives and 3 criteria. Fig. 6 depicts the division of the base-plane by the separating planes which yields the maximum number of feasible rankings. There are six separating planes which are labeled as: L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, and L_{34}. First L_{12}, L_{13}, and L_{14} are added onto the base-plane. Since these three lines are in the general position, the base-plane is divided into seven regions. Next, L_{23} is placed onto the base-plane. It can be seen that L_{23} must cross the intersection of L_{12}, and L_{13}. Hence, L_{23} can only divide the base-plane into three new regions. Similarly, L_{24} can only divide the base-plane into four new regions. Finally, when L_{34} is added, it must cross both the intersections of L_{13}, L_{14}, and L_{23}, L_{24}. Hence it can only divide the base-plane into four new regions. When all the separating planes are added onto the base-plane, it is divided into $(7 + 3 + 4 + 4 = 18)$

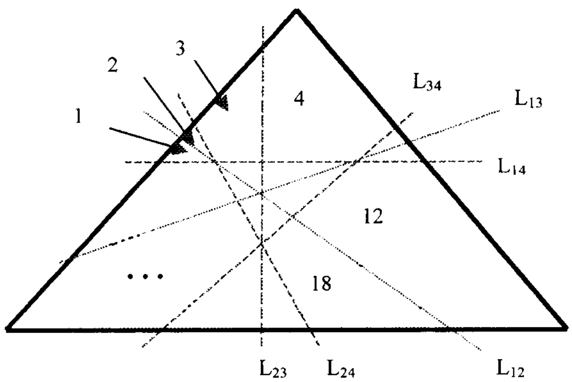

Fig. 6. Division of the base-plane with 3 criteria and 4 alternatives.

ranking regions. This is exactly the same number obtained from Eq. (17).

3.4. Partition of spaces when $n \geqslant 4$

In order to calculate the value of $F_n(m)$ when $n \geqslant 4$ we follow exactly the same procedure proposed in the previous sub-section. All the k separating planes will be added onto the base-plane one by one according to the sequence: $(L_{12}, L_{13}, L_{14}, \dots, L_{1m})$, $(L_{23}, L_{24}, \dots L_{2m})$, $(L_{34}, L_{35}, \dots, L_{3m}), \dots, L_{(m-1)m}$. When L_{ij} $(i = 2, 3, \dots, m-1$ and $j = i+1, i+2, \dots, m)$ is placed onto the base-plane, the "old" separating planes that are already on the base-plane will cross the newly added separating plane at $(K_{ij} - (i-1))$ intersections. These intersections are $(n-3)$-dimensional hyperplanes.

For $n = 3$, these intersections are points and points are always in the general position. However, as the dimension of the space increases, the problem becomes more complex. For $n \geqslant 4$, these intersection hyperplanes can no longer be in the general position on L_{ij}. (This has been proved by the authors but is omitted in this paper.) Therefore Eq. (16) cannot be used to calculate $F_n(m)$ when $n \geqslant 4$ and there is no closed-form formula for $F_n(m)$ if $n \geqslant 4$. Thus, in order to calculate the value of $F_n(m)$, some recursive functions have been derived.

Let Z_{ij} be the number of hyperplanes that cannot be in the general position among the $(K_{ij} - (i-1))$ intersections on L_{ij}. Let $X_{ij} = K_{ij} - i + 1 - Z_{ij}$. From the definition of Z_{ij}, the variable X_{ij} denotes the number of hyperplanes that could be in the general position among these $(K_{ij} - i + 1)$ hyperplanes. Let $f_{n-1}(x, y)$ be the number of regions that x $(n-1)$-dimension hyperplanes can divide an n-dimension base-plane into. We also define that among the x hyperplanes, y of them cannot be in the general position on the base-plane. Thus, for $n \geqslant 4$, the following functions have been derived (details of how these equations are derived is available from the first author):

$$F_n(m) = h_{n-1}(m-1) + \sum_{i=2}^{m-1} \sum_{j=i+1}^{m} f_{n-2}(K_{ij} - i + 1,\ Z_{ij}), \tag{18}$$

$$f_{n-2}(K_{ij} - i + 1,\ Z_{ij}) = h_{n-2}(X_{ij}) + \sum_{l=0}^{Z_{ij}-1} f_{n-3}(X_{ij} - i + l,\ l), \tag{19}$$

where

$$f_1(K, l) = h_1(K),$$

$$X_{ij} = K_{ij} - i + 1 - Z_{ij},$$

and

$$f_N(K, 0) = h_N(K). \tag{20}$$

In Eqs. (18) and (19) the values of different Z_{ij} can be obtained (see also Table 6) by observing the intersections of L_{ij} and those separating planes that are added onto the base-plane before L_{ij}.

Please note that Eq. (19) denotes a recursive function. When this equation is solved, the value of the function $f_{n-2}(K_{ij} - i + 1, Z_{ij})$ is obtained. Next, Eq. (18) can be applied to calculate $F_n(m)$. In this way, one can calculate the maximum number of ranking sequences in an MCDM problem with m alternatives and n criteria. Table 7 lists some of the values of $F_n(m)$ for different m and n values. These results demonstrate an interesting property of the MCDM problems. That is, the number of decision criteria may greatly influence the number of feasible rankings in an MCDM problem.

Table 6
Values of Z_{ij} for different i and $j(= i, i+1, \ldots, m)$ values

i Values										
1		2		3		4		...	$m-1$	
i	Z_{ij}	i	Z_{ij}	i	Z_{ij}	i	Z_{ij}	...	i	Z_{ij}
2	0	3	0	4	$m-3$	5	$2m-7$		m	$(m-2)(m-3)/2$
3	0	4	1	5	$m-2$	6	$2m-6$			
...	0	...		...		...		...		
m	0	m	$m-3$	m	$2m-7$	m	$3m-12$			

Table 7
Some values of $F_n(m)$ (*maximum number of feasible rankings*) for different n and m values

No. of Alts, m	Number of criteria, n						
	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2
3	1	4	6	6	6	6	6
4	1	7	18	24	24	24	24
5	1	11	46	94	120	120	120
6	1	16	101	338	681	720	720
7	1	22	197	980	3047	3833	5040

4. Some application issues

This paper demonstrates that when the number of criteria is less than the number of alternatives, we can never get all the possible rankings of the alternatives simply by changing the criteria weights. What is more, the possible number of rankings that can be generated may be significantly less than its upper bound: the factorial of the number of the alternatives. Let R be defined as the ratio $F_n(m)/m!$. To compare this maximum number of possible ranking sequences for different numbers of alternatives, the ratio R is used as the measure of the relative value of this maximum number of feasible rankings. Fig. 7 depicts for each specific number of alternatives how this ratio R changes in terms of the number of criteria. These results were derived from the results given in Table 7.

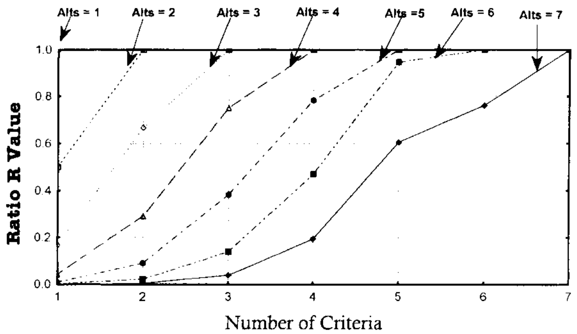

Fig. 7. Relative value of the maximum number of feasible ranking sequences.

In Fig. 7, for a given number of decision criteria, the smaller the number of alternatives is, the higher the value of R is. As the number of alternatives increases, the value of R drops dramatically. Also, for a given number of alternatives, as the number of decision criteria increases, the value of R increases with it until the number of criteria is equal to the number of alternatives, in which case R reaches its maximum value of 1. After this point, the values of R will not change any more even if the number of criteria still increases. When the number of criteria increases, the smaller the number of alternatives is, the faster the value of R increases.

Fig. 7 indicates the relationship between the number of decision criteria, the number of alternatives and the maximum number of feasible rankings of the decision making model. This finding can make contributions to both the theory and application of multi criteria decision making. One of the most important tasks in MCDM is the choice of the criteria. Keeney and Raiffa (1976) stated that the desirable properties of a set of criteria should be: (i) completeness; (ii) operational; (iii) decomposable; (iv) non-redundancy; and (v) to be of minimum size. The study of the maximum number of feasible rankings suggests that the ratio R may serve as a quantitative measure of whether a given number of criteria is discriminating enough among the alternatives.

5. Conclusions

This paper discussed the problem of at most how many different ranking sequences can exist for a given deterministic MCDM problem. This study showed that the maximum number of feasible ranking sequences for a deterministic MCDM problem is a function of both the number of alternatives and the number of criteria. When there are less criteria than alternatives, then this maximum number can be much smaller than the factorial of the alternatives.

For decision matrices with less than four criteria, there exists a closed-form formula of this maximum number of terms of the number of criteria and the number of alternatives. For a problem with more than four criteria, however, such a closed-form formula does not exist. Instead, a decomposition method is developed to calculate this number. The results of this research may provide a theoretical foundation for a decision maker to measure the completeness of his/her choice of the set of criteria.

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References

Brualdi, R.A., 1979. Introductory Combinatorics. North-Holland, Oxford.

Chen, S.-J., Hwang, C.-L., 1992. Fuzzy multiple attributes decision making. Lecture Notes in Economics and Mathematical Systems, No. 375, Springer, New York.

Fishburn, P.C., 1967. Additive Utilities with Incomplete Product Set: Applications to Priorities and Assignments. ORSA Publications, Baltimore, MD.

Keeney, R.L., Raiffa, H., 1976. Decisions with Multiple Objectives: Preferences and Value Tradeoffs. Wiley, New York.

Roman, S., 1987. An Introduction to Linear Algebra, second ed. HBJ Publishers, Orlando, FL.

Saaty, T.L., 1994. Fundamentals of decision making and priority theory with the AHP. RWS Publications, Pittsburgh, PA.

Triantaphyllou, E., Mann, S.H., 1989. An examination of the effectiveness of multi-dimensional decision-making methods: A decision-making paradox. Decision Support Systems 5, 303–312.