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A Minimization Approach to Membership Evaluation in Fuzzy Sets and Error Analysis

E. TRIANTAPHYLLOU,[1] P. M. PARDALOS,[2] AND S. H. MANN[3]

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Abstract. Evaluation of the degree of membership in fuzzy sets is a fundamental topic in fuzzy set theory. Saaty (Ref. 1) proposes a method for solving this problem that has been widely accepted. In this paper, we examine the problem from an error minimization point of view that attempts to reflect the real intentions of the decision maker. When this approach is used, the findings reveal that fuzzy sets of different cardinalities have dramatically different requirements in the consistency level of the input data as far as the error minimization criterion is concerned.

Key Words. Fuzzy sets, degree of membership, eigenvectors, least-square problems, decision making.

1. Introduction

Values between 0 and 1 are used to determine the degree of membership of the elements of a fuzzy set. The degrees of membership are supposed to be a good model of the way people perceive categories (Ref. 2). Usually, the most representative members in a fuzzy set are assigned to the value of 1.00 and nonmembers to the value of 0.00. Then, the main problem is to determine the degree of membership (i.e., a number between 0 and 1) of the between members. Psychologists (Ref. 3) have found that people can easily identify representative members in a fuzzy set, while they have difficulties in identifying the other members. The importance of evaluating

[1] PhD Candidate in Operations Research and in Industrial and Management Systems Engineering, Department of Industrial and Management Systems Engineering, Pennsylvania State University, University Park, Pennsylvania.
[2] Assistant Professor, Department of Computer Science, Pennsylvania State University, University Park, Pennsylvania.
[3] Professor of Operations Research, School of Hotel, Restaurant, and Institutional Management, Pennsylvania State University, University Park, Pennsylvania.

the membership degrees in applications of fuzzy set theory in engineering and scientific fields is best illustrated in the 1,800 references given by Gupta *et al.* (Ref. 4).

Saaty (Refs. 1 and 5) has suggested a possible solution to the membership evaluation problem based on eigenvalue theory. In both Chu *et al.* (Ref. 6) and Federov *et al.* (Ref. 7), Saaty's method has been viewed as a modified least-square problem. In this paper, a least-square approach is used on the data derived from the pairwise comparisons as proposed by Saaty. However, the present approach uses an error minimization function that attempts to reflect the real intentions of the decision maker while making the pairwise comparisons. The findings of this paper demonstrate that the number of elements in a fuzzy set is very critical in determining the impact of the consistency of the pairwise comparisons.

2. Literature Review

Let $A_1, A_2, \ldots, A_n$ be the members of a fuzzy set. We are interested in evaluating the membership values of the above members. Saaty (Refs. 1 and 5) proposes to use a matrix A of rational numbers taken from the set $\{1/9, 1/8, 1/7, \ldots, 1, 2, \ldots 7, 8, 9\}$. Each entry of the above matrix A represents a pairwise judgement. Specifically, the entry a_{ij} denotes the number that estimates the relative membership of element A_i when it is compared with element A_j. Obviously, $a_{ij} = 1/a_{ji}$ and $a_{ii} = 1$. That is, the matrix is a reciprocal one.

Let us first examine the case in which it is possible to have perfect values a_{ij}. In this case, it is $a_{ij} = W_i/W_j$ (W_s denotes the actual value of the element s) and the previous reciprocal matrix A is consistent, that is,

$$a_{ij} = a_{ik}a_{kj}, \qquad i, j, k = 1, 2, 3, \ldots, \tag{1}$$

where n is the number of elements in the fuzzy set.

It can be proved that A has rank 1, with $\lambda = n$ to be its nonzero eigenvalue. Then, we have

$$Ax = nx, \tag{2}$$

where x is an eigenvector. From the fact that $a_{ij} = W_i/W_j$, the following relations are obtained:

$$\sum_{j=1}^{n} a_{ij}W_j = \sum_{j=1}^{n} W_i = nW_i, \qquad i = 1, 2, 3, \ldots, n, \tag{3}$$

or

$$AW = nW. \tag{4}$$

Equation (4) states that n is an eigenvalue of A with W a corresponding eigenvector. The same equation also states that, in the perfectly consistent case (i.e., $a_{ij} = a_{ik}a_{kj}$), the vector W, with the membership values of the elements $1, 2, 3, \ldots, n$, is the principal right-eigenvector (after normalization) of matrix A.

In the nonconsistent case (which is the most common), the pairwise comparisons are not perfect, that is, the entry a_{ji} might deviate from the real ratio W_i/W_j (i.e., from the ratio of the real membership values W_i and W_j). In this case, the previous expression (1) does not hold for all the possible combinations. Now, the new matrix A can be considered as a perturbation of the previous consistent case. When the entries a_{ij} change slightly, then the eigenvalues change in a similar fashion (Ref. 5). Moreover, the maximum eigenvalue is close to n (greater than n), while the remaining eigenvalues are close to zero. Thus, in order to find the membership values in the nonconsistent cases, one should find an eigenvector that corresponds to the maximum eigenvalue λ_{max}. That is to say, one should find the principal right-eigenvector W that satisfies

$$AW = \lambda_{max} W, \qquad \text{where} \quad \lambda_{max} \approx n.$$

Saaty estimates the reciprocal right-eigenvector W by multiplying the entries in each row of the matrix A together and taking the nth root (n is the number of the elements in the fuzzy set). Since we desire to have values that add up to 1.00, we normalize the previously found vector by the sum of the above values. If we want to have the element with the highest value to have a membership value equal to 1.00, we divide the previously found vector by the highest value.

Under the assumption of total consistency, if the judgments are gamma distributed (something that Saaty claims is the case), the principal right-eigenvector of the resultant reciprocal matrix A is Dirichlet distributed. If the assumption of total consistency is relaxed, then Vargas (Ref. 8) proves that the hypothesis that the principal right-eigenvector follows a Dirichlet distribution is accepted if the consistency ratio is 0.10 or less.

The consistency ratio (CR) is obtained by first estimating λ_{max}. Saaty estimates λ_{max} by adding the columns of matrix A and then multiplying the resulting vector with the vector W. Then, he uses what he calls the consistency index (CI) of the matrix A. He defines CI as follows:

$$\text{CI} = (\lambda_{max} - n)/(n - 1).$$

Then, the consistency ratio CR is obtained by dividing the CI by the random consistency index (RCI) as given in Table 1.

Each RCI is an average random consistency index derived from a sample of size 500 of randomly generated reciprocal matrices with entries from the set $\{1/9, 1/8, 1/7, \ldots, 1, 2, \ldots, 7, 8, 9\}$ to see if its CI is 0.10 or less.

Table 1. RCI values of sets of different order n.

n	1	2	3	4	5	6	7	8	9
RCI	0	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45

If the previous approach yields a CR greater than 0.10, then a reexamination of the pairwise judgments is recommended until a CR less than or equal to 0.10 is achieved.

Chu *et al.* (Ref. 6) observed that, given the data a_{ij}, the values W_i to be estimated are desired to have the following property:

$$a_{ij} \approx W_i / W_j. \tag{5}$$

This is true, since a_{ij} is meant to be the estimate of the ratio W_i/W_j. Then, in order to get the estimates for the W_i given the data a_{ij}, they propose the following constrainted optimization problem:

$$\min S = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} W_j - W_i)^2, \tag{6}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} W_i = 1,$$

$$W_i > 0, \qquad i = 1, 2, 3, \ldots, n.$$

They also give an alternative expression S_1 that is more difficult to solve numerically. That is,

$$S_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - W_j / W_i)^2. \tag{7}$$

In Federov *et al.* (Ref. 7), a variation of the above least-square formulation is proposed. For the case of only one decision maker, they recommend the following models:

$$\log a_{ij} = \log W_i - \log W_j + \Psi_1(W_i, W_j)\varepsilon_{ij} \tag{8}$$

and

$$a_{ij} = W_i / W_j + \Psi_2(W_i, W_j)\varepsilon_{ij}, \tag{9}$$

where W_i and W_j are the true (and unknown) membership values; $\Psi_1(X, Z)$ and $\Psi_2(X, Z)$ are given positive functions (when $X, Z > 0$). The random errors ε_{ij} are assumed independent with zero mean and unit variance. Using these two assumptions, they are able to calculate the variance of each individual estimated membership value. However, they fail to give a way of selecting the appropriate positive functions. In Example 3.2, presented later, a sample problem that originates in Ref. 1 and later in Ref. 7 is solved for different functions Ψ_1, Ψ_2 using the Federov method.

3. Considering the Human Rationality Factor

According to the human rationality assumption, the decision maker is a rational person. Rational persons are defined here as individuals who try to minimize their regret (Ref. 9), to minimize losses, or to maximize profit (Ref. 10). In the membership evaluation problem, minimization of regret, losses, or maximation of profit could be interpreted as the effort of the decision maker to minimize the errors involved in the pairwise comparisons.

As it is stated in previous paragraphs, in the inconsistent case, the entry a_{ij} of the matrix A is an estimate of the real ratio W_i/W_j. Since it is an estimate, the following is true:

$$a_{ij} = (W_i/W_j)d_{ij}, \qquad i, j = 1, 2, 3, \ldots, n. \tag{10}$$

In the above relation, d_{ij} denotes the deviation of a_{ij} from being an accurate judgment. Obviously, if $d_{ij} = 1$, then the a_{ij} was perfectly estimated. From the previous formulation, we conclude that the errors involved in these pairwise comparisons are given by

$$\epsilon_{ij} = d_{ij} - 1.00,$$

or using (10) above,

$$\epsilon_{ij} = a_{ij}(W_j/W_i) - 1.00. \tag{11}$$

When a fuzzy set contains n elements, then Saaty's method requires the estimation of the following $n(n-1)/2$ pairwise comparisons:

$$W_2/W_1,\ W_3/W_1,\ W_4/W_1, \ldots, W_n/W_1, \tag{12-1}$$

$$W_3/W_2,\ W_4/W_2, \ldots, W_n/W_2, \tag{12-2}$$

$$W_4/W_3, \ldots, W_n/W_3, \tag{12-3}$$

$$\vdots$$

$$W_{n-1}/W_n. \tag{12-n}$$

The corresponding $n(n-1)/2$ errors are [using relations (11) and (12)]:

$$\epsilon_{ij} = a_{ij}(W_j/W_i) - 1.00, \qquad i, j = 1, 2, \ldots, n, j > 1. \tag{13}$$

Since the W_i's are degrees of membership that add up to 1.00, the following relation should also be satisfied:

$$\sum_{i=1}^{n} W_i = 1.00. \tag{14}$$

Apparently, since the W_i's represent degrees of membership, we also have

$$W_i > 0, \qquad i = 1, 2, 3, \ldots, n.$$

Relations (13) and (14), when the data are consistent (i.e., all the errors equal to zero), can be written as follows:

$$BW = b. \tag{15}$$

The vector b has zero entries everywhere, except that the last one is equal to 1.00; the matrix B has the following form (blank entries represent zeros):

$$B = \left[\begin{array}{ccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & n-1 & n & & \\
-1 & a_{1,2} & & & & & & & & & & 1 \\
-1 & & a_{1,3} & & & & & & & & & 2 \\
-1 & & & a_{1,4} & & & & & & & & 3 \\
\vdots & & & & \ddots & & & & & & & \vdots \\
 & & & & & & & & a_{1,n-1} & & & \\
-1 & & & & & & & & & a_{1,n} & & n-1 \\
 & -1 & a_{2,3} & & & & & & & & & 1 \\
 & -1 & & a_{2,4} & & & & & & & & 2 \\
 & -1 & & & a_{2,5} & & & & & & & 3 \\
 & \vdots & & & & \ddots & & & & & & \vdots \\
 & & & & & & & & a_{2,n-1} & & & \\
 & -1 & & & & & & & & a_{2,n} & & n-2 \\
 & & & & & & & & & & & \vdots \\
 & & & & & & & & -1 & a_{n-1,n} & & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & &
\end{array}\right]$$

The error minimization issue is interpreted in many cases (regression analysis, linear least-square problem) as the minimization of the sum of squares of the residual vector $r = b - BW$ (Ref. 11). In terms of the formulation (15), this means that, in a real-life situation (i.e., errors are not zero), any more), the real intention of the decision maker is to minimize the expression

$$f^2(x) = \|b - BW\|_2^2, \tag{16}$$

which is a typical linear least-square problem.

If we use the notation decribed previously, then the quantity (6) that is minimized in Ref. 6 becomes

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} W_j - W_i)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (\epsilon_{ij} W_i)^2,$$

and the alternative expression (7) becomes

$$S_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - W_j / W_i)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (\epsilon_{ij} W_i / W_j)^2.$$

Clearly, both expressions are too complicated to reflect in a reasonable way the intentions of the decision maker.

The models proposed by Federov *et al.* (Ref. 7) are closer to the one developed under the human rationality assumption. The only difference is that, instead of the relations

$$\log a_{ij} = \log W_i - \log W_j + \Psi_1(W_i, W_j)\epsilon_{ij}$$

and

$$a_{ij} = W_i / W_j + \Psi_2(W_i, W_j)\epsilon_{ij},$$

the following simpler expression is used:

$$a_{ij} = (W_i / W_j) d_{ij}$$

or

$$a_{ij} = (W_i / W_j)(\epsilon_{ij} + 1.00). \tag{17}$$

However, as Example 3.2 illustrates, the performance of this method is greatly dependent on the selection of the functions $\Psi_1(X, Z)$ or $\Psi_2(X, Z)$ and now these functions are further modified by (17).

Example 3.1. Let us assume that the following is the matrix of pairwise comparisons for a set of four elements:

$$A = \begin{bmatrix} 1 & 2/1 & 1/5 & 1/9 \\ 1/2 & 1 & 1/8 & 1/9 \\ 5/1 & 8/1 & 1 & 1/4 \\ 9/1 & 9/1 & 4/1 & 1 \end{bmatrix}.$$

Using the methods presented in previous sections, we can see that

$$\lambda_{max} = 4.226,$$

$$CI = (4.226 - 4)/(4 - 1) = 0.0753,$$

$$CR = CI/0.90 = 0.0837 < 0.10.$$

The formulation (15) that corresponds to this example is

$$\begin{bmatrix} -1 & 2/1 & 0.0 & 0 \\ -1 & 0.0 & 1/5 & 0 \\ -1 & 0.0 & 0 & 1/9 \\ 0.0 & -1 & 1/8 & 0 \\ 0.0 & -1 & 0 & 1/9 \\ 0.0 & 0.0 & -1 & 1/4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.0 \end{bmatrix}.$$

The vector V that solves the above least-square problem is calculated to be

$$V = (0.065841, 0.039398, 0.186926, 0.704808);$$

the sum of squares of the residual vector components is 0.003030. The average squared residual for this problem is $0.003030/((4 \times (4-1)/2)+1 = 0.000433$; that is, the average residual is $\sqrt{(0.000433)} = 0.020806$.

Example 3.2. The second example uses the same data used originally in the paper by Saaty (Ref. 1) and later in the two papers by Chu *et al.* and Federov *et al.* (Refs. 6 and 7). These data are presented in Table 2.

Table 3 presents a summary of the results (as found in the corresponding references) when the methods described in Section 2 are used. The power method for deriving the eigenvector was applied as presented by Kalaba

Table 2. Data for Example 3.2.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1)	1	4	9	6	6	5	5
(2)	1/4	1	7	5	5	3	4
(3)	1/9	1/7	1	1/5	1/5	1/7	1/5
(4)	1/6	1/5	5	1	1	1/3	1/3
(5)	1/6	1/5	5	1	1	1/3	1/3
(6)	1/5	1/3	7	3	3	1	2
(7)	1/5	1/4	5	3	3	1/2	1

Table 3. Comparison of the membership values for the Data in Table 2.

Method used	Element in set (1)	(2)	(3)	(4)	(5)	(6)	(7)	Average residual
Saaty's eigenvector method	0.429	0.231	0.021	0.053	0.053	0.119	0.095	0.134
Power method eigenvector	0.427	0.230	0.021	0.052	0.052	0.123	0.094	0.135
Chu's method	0.487	0.175	0.030	0.059	0.059	0.104	0.085	0.097
Federov Model 1 with $\Psi_1 = 1$	0.422	0.232	0.021	0.052	0.052	0.127	0.094	0.138
Federov Model 2 with $\Psi_2 = 1$	0.386	0.287	0.042	0.061	0.061	0.088	0.075	0.161
Federov Model 2 with $\Psi_2 = \lvert W_i - W_j \rvert$	0.383	0.262	0.032	0.059	0.059	0.122	0.083	0.152
Federov Model 2 with $\Psi_2 = (W_i / W_j)$	0.047	0.229	0.021	0.051	0.051	0.120	0.081	0.130
Least-square method under HR assumption	0.408	0.147	0.037	0.054	0.054	0.080	0.066	0.082

et al. (Ref. 12). The last row of Table 3 shows the results obtained by using the least-square method under the human rationality assumption (HR).

As is shown in the last column of Table 3, the performance of each method is very different as far the mean residual is concerned. The results also illustrate how critical is the role of the functions $\Psi_1(X, Z)$ and $\Psi_2(X, Z)$ in the Federov *et al.* method. The mean residual obtained by using the least-square method under the human rationality assumption is the smallest one by 16%.

4. Average Error per Comparison for Fuzzy Sets of Different Order

In this section, we generate random reciprocal matrices of pairwise comparisons of different order and consistency index (CI). For each test problem, the least-square problem, as derived under the human rationality assumption, is solved and the average residual is recorded. The same problem is also solved by using the eigenvalue method. The average residual is considered here as an indicator of the effectiveness in estimating successfully membership degrees from a set of pairwise comparisons. The simulation program was written in FORTRAN and the resulting least-square problems were solved using the appropriate IMSL subroutines. The results are presented in Table 4 and depicted in Figs. 1 and 2.

Table 4. Average residual and CI versus order of fuzzy set and CR when the human rationality assumption (HR) and the eigenvalue method (EM) is used; results correspond to one hundred observations.

Order of set		Corresponding CR coefficient								
		0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
7	CI	0.0264	0.0396	0.0528	0.0660	0.0792	0.0924	0.1056	0.1188	0.1320
	EM	(*)	(*)	(*)	(*)	0.6446	0.6674	0.8497	0.8930	0.9414
	HR	(*)	(*)	(*)	(*)	0.5094	0.5279	0.5560	0.6024	0.6112
6	CI	0.0248	0.0372	0.0496	0.0620	0.0744	0.0868	0.0992	0.1116	0.1240
	EM	0.4049	0.5546	0.6336	0.7100	0.7798	0.8369	0.9347	0.9681	1.0554
	HR	0.2982	0.4035	0.4467	0.4615	0.5240	0.5506	0.5956	0.6127	0.6503
5	CI	0.0224	0.0336	0.0448	0.0560	0.0672	0.0784	0.0896	0.1008	0.1120
	EM	0.4957	0.6192	0.7598	0.7927	0.8682	0.9769	1.0303	1.0984	1.2097
	HR	0.3382	0.4276	0.4912	0.4934	0.5575	0.6030	0.6334	0.6742	0.6913
4	CI	0.0180	0.0270	0.0360	0.0450	0.0540	0.0630	0.0720	0.0810	0.0900
	EM	0.5420	0.6629	0.7787	0.8534	0.9779	1.0696	1.1388	1.2451	1.2759
	HR	0.3509	0.4193	0.4883	0.5284	0.5777	0.6323	0.6807	0.7252	0.7340
3	CI	0.0116	0.0174	0.0232	0.0290	0.0348	0.0406	0.0464	0.0522	0.0580
	EM	0.5005	0.6690	0.7847	0.8196	0.9979	1.0486	1.1012	1.2225	1.3060
	HR	0.2869	0.3554	0.3836	0.4785	0.4998	0.5535	0.6190	0.6215	0.6449

(*) Values were not found for these cells due to the expensive requirement for CPU time.

For each case, the number of the generated random matrices was varying but large enough to ensure that the means converged to within a small error tolerance. It was observed that, for large matrices with small CI values, the sample size could be small (less than 100) and still achieve this similar convergence. However, the opposite is true for small matrices with large CI values. In Figs. 1 and 2, the vertical axis represents the average residual, while the horizontal one represents the consistency index (CI). The CI was selected such that the CR would be 0.02, 0.03, 0.04, ..., 0.10. Due to the difficulty in obtaining large random matrices when the CI (or the corresponding CR) is very small, the supercomputer facilities at Cornell University were used. Figure 1 depicts average residuals when the human rationality assumption is used, while Fig. 2 depicts residuals when the eigenvector method is used.

As the plots in Figs. 1 and 2 illustrate, the average residual is a function of both the consistency index of the data as well as the order of the input matrix (i.e., the number of elements in the fuzzy set under consideration). Regression analyses suggest that, for fuzzy sets of a given order, the average residual is linearly related to the consistency index. As the CI of the data decreases, so does the average residual. This is expected because, as the CI

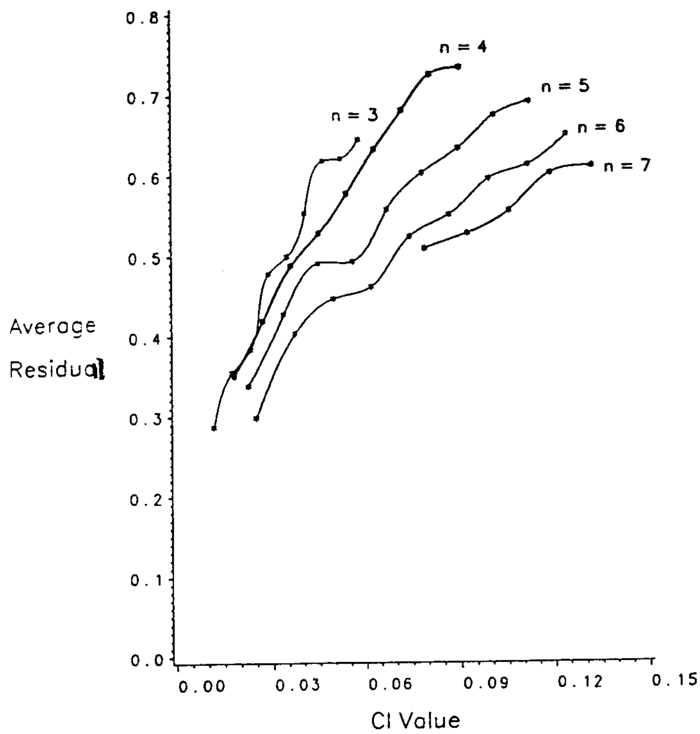

Fig. 1. Average residual and CI versus order of fuzzy set when the human rationality assumption is used; results correspond to one hundred observations.

reaches zero, the input data tend to be perfectly consistent. From these results, it can be seen that the average residuals are significantly smaller when the method that is based on the human rationality assumption is used. The same plots also illustrate the importance of the order of a fuzzy set. For small sets, the CI has to be small in order to keep the average error at a low level. However, as the size of fuzzy sets increases, the CI can be relatively larger and still achieve small average error per comparison. That is, the input data in the case of large fuzzy sets can be more inconsistent than in the case of smaller fuzzy sets, with respect to the error per comparison issue.

5. Conclusions

The Saaty method of evaluating the membership degrees in fuzzy sets has captured the interest of many researchers. This is mainly due to the

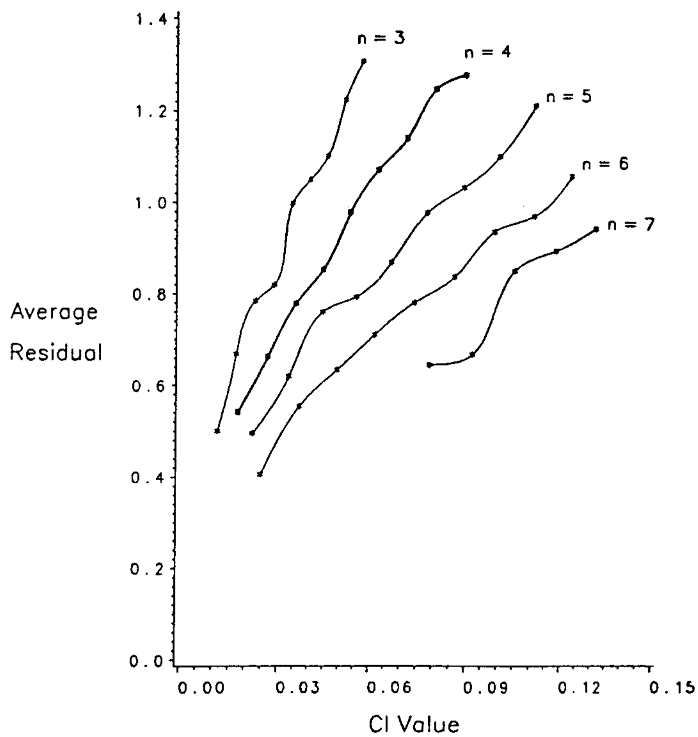

Fig. 2. Average residual and CI versus order of fuzzy set when the eigenvalue method is used; results correspond to one hundred observations.

mathematical properties of the method and the fact that the input data are rather easy to obtain.

However, as the findings of the present paper demonstrate, the analysis of the errors reveals a new dimension of the membership value problem. Although the CR value can be kept to less than 0.10 (and hence ensure satisfaction of the Dirichlet distribution criterion as described previously), the mean residual can vary significantly. The results demonstrate that, even with data that yield CR less than 0.10, the CR (or the correspodning CI) has to be kept at low levels for small fuzzy sets and at somewhat higher levels for large fuzzy sets, even when the method that is based on the human rationality assumption is used. As is demonatrated by Triantaphyliou *et al.* (Ref. 13), small changes in the membership values can mean the difference between selecting one alternative instead of another in many decision-making problems. Since the role of membership values is crucial in many real-life problems, further understanding of these fuzzy set problems is critical.

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