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A sensitivity analysis of a (t_i, S_i) inventory policy with increasing demand

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The inventory model with a (t_i, S_i) policy with increasing demand is common in many practical applications. Under this model the inventory system operates only during a prescribed period of time. It also corresponds to the classical deterministic, stationary demand model with no shortages. This present sensitivity analysis reveals that there is a strong relationship among the optimal number of replenishments, the total demand, the planning horizon, and the optimal cost.

deterministic models; parametric analysis; planning horizons

1. Introduction

Consider an inventory system in which the demand is given by the following relation:

$$r(t) = a_0 + a_1(t). \tag{0}$$

This demand pattern captures the essence of a product's demand during its life cycle. This paper studies the sensitivity of the (t_i, S_i) inventory policy with increasing demand. In this policy t_i denotes the scheduling periods and S_i the order levels. The sensitivity analysis is done with respect to the optimal total cost, the optimal number of replenishments, the time horizon, and the demand assumed during this horizon. The present inventory system is assumed to have the following characteristics [4]:

1. The system operates only during a prescribed period which is H units of time long.
2. The demand is continuous and increases linearly with time at rate $r(t)$. In this situation the parameters a_0 and a_1 of (0) are $a_0 = 0$ and $a_1 > 0$. Therefore, the demand pattern for this system is

   $$r(t) = a_1 * t. \tag{1}$$

   (The ideas can also be extended to linearly decreasing demand.)
3. During the period H there exists a total demand for D quantity units. That is, the constant a_1 in (1) can be calculated as follows:

   $$D = \int_0^H r(t)\,\mathrm{d}t = \int_0^H a_1 t\,\mathrm{d}t = \tfrac{1}{2}aH^2. \tag{2}$$

   Hence

   $$r(t) = \left(\frac{2D}{H^2}\right)t. \tag{3}$$

4. The only relevant unit costs are c_1, the unit carrying cost in \$ per item per unit time, and c_2, the unit replenishment cost in \$ per run.

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As stated in [1–8], under the present policy the total amount in inventory J_1, during the period H can be found as follows. Let $I(\tau)$ represent the inventory at time τ. Then, assuming zero lead times and zero reorder points,

$$J_1 = \sum_{i=1}^{m} \int_{T_{i-1}}^{T_i} I(\tau)\, \mathrm{d}r$$
$$= \sum_{i=1}^{m} \int_{T_{i-1}}^{T_i} \int_{t=T_{i-1}}^{\tau} I\{\mathrm{d}t\}\, \mathrm{d}\tau \quad [\text{where } I\{\mathrm{d}t\} = (\mathrm{d}[I(t)]/\mathrm{d}t) * \mathrm{d}t]$$
$$= \sum_{i=1}^{m} \int_{T_{i-1}}^{T_i} (t - T_{i-1}) r(t)\, \mathrm{d}t,$$

which leads to

$$J_1 = \tfrac{2}{3}DH - \frac{D}{H^2} \sum_{i=1}^{m} (T_i^2 - T_{i-1}^2) T_{i-1} \tag{4}$$

where m is the number of replenishments and T_i are the scheduling points, and where $T_0 = 0$, $T_{i=1} \leqslant T_i$ and $T_m = H$.

Then, the total relevant cost per unit time for this inventory system is given by (5)

$$C(m, T_1, T_2, T_3, \ldots, T_{m-1}) = \frac{c_1 J_1}{H} + \frac{c_2 m}{H} \tag{5}$$

In the previous equation it is assumed that the time horizon H and the interest rates are such that discounting is unimportant. Relations (4) and (5) combined yield

$$C(m, T_1, T_2, T_3, \ldots, T_{m-1}) = \frac{c_1}{H}\left[\tfrac{2}{3}DH - \frac{D}{H^2} \sum_{i=1}^{m} (T_i^2 - T_{i-1}^2) T_{i-1}\right] + \frac{c_2 m}{H}. \tag{6}$$

The optimal values for m, T_1, T_2, $T_3, \ldots, T_{m-1}$ which minimize the total cost can be found by differentiating with respect to each of the T_i variables. Then, it turns out that the optimal values for the T_i variables are given by the recursive relation

$$T_i = \alpha_i T_1 \quad \text{for } i = 0, 1, 2, \ldots, m-1, \text{ and } T_m = H = \alpha_m T_1 \tag{7}$$

where the α_i's are derived using the recursive relationship

$$\alpha_0 = 0, \qquad \alpha_1 = 1, \text{ and } \alpha_j = \sqrt{3\alpha_{j-1}^2 - \alpha_{j-1}\alpha_{j-2}} \quad \text{for } j = 2, 3, 4, \ldots, m. \tag{8}$$

From the relation $T_i = \alpha_i T_1$ it can be seen that the quantities α_i represent the ratio of the scheduling point T_i when it is compared with T_1 (for $i = 0, 1, 2, \ldots, m$).

The optimal value of m (i.e., the optimal number of replenishments in period H) can be found by first combining (6) and (7)

$$C(m, T_1, T_2, T_3, \ldots, T_{m-1}) = C(m) = c_1 D\left[\tfrac{2}{3} - \beta_m\right] + \frac{c_2 m}{H} \tag{9}$$

where

$$\beta_m = \frac{1}{\alpha_m^3} \sum_{i=1}^{m} (\alpha_i^2 - \alpha_{i-1}^2) \alpha_{i-1}. \tag{10}$$

The optimal value of m, m_0, can be obtained by observing that the following two conditions should be satisfied:

$$c(m_0) \leqslant c(m_0 + 1) \text{ and } c(m_0) \leqslant c(m_0 - 1).$$

When the previous two conditions are combined with (9), they yield

$$\Delta\beta_{m_0+1} \leqslant \frac{c_2}{c_1 DH} \leqslant \Delta\beta_{m_0} \tag{11}$$

where $\Delta\beta_{m_0} = \beta_{m_0} - \beta_{m_0-1}$, $\Delta\beta_{m_0+1} = \beta_{m_0+1} - \beta_{m_0}$ and the β_{m_0}, β_{m_0-1}, β_{m_0} can be found using (10).

From the previous considerations it follows that the optimal cost c_0, and the optimal number of replenishments m_0, can be calculated in two steps:

Step 1. Find the integer m_0, that satisfies the relation

$$\Delta\beta_{m_0+1} \leqslant \frac{c_2}{c_1 DH} \leqslant \Delta\beta_{m_0} \tag{12}$$

where the $\Delta\beta_i$'s can be calculated using (8), (10) and (11).

Step 2. Find the optimal cost c_0 using the formula

$$c_0 = c(m_0) = c_1 D\left[\tfrac{2}{3} - \beta_{m_0}\right] + \frac{c_2 m_0}{H}. \tag{13}$$

2. Transformations

Let $W_1 = c_1 D$ and $W_2 = c_2/H$. Then, (12) and (13) become (14) and (15), respectively.

$$\Delta\beta m_0 + 1 \leqslant \frac{W_2}{W_1} \leqslant \Delta\beta_{m_0} \tag{14}$$

and

$$c_0 = w_1\left[\tfrac{2}{3} - \beta_{m_0}\right] + W_2 m_0. \tag{15}$$

If the two parameters W_1 and W_2 are given then, using (8), (10), (14), and (15), the value of the integer m_0 and the optimal cost c_0 can be determined uniquely. Very often in practice the costs c_1 and c_2 can be considered as constants, while the demand D and horizon H vary. When this is true we can talk about W_1 changes and mean changes of the demand D or W_2 changes and mean changes of the horizon H. However, we use W_1 and W_2 for generality. A computer program was written that calculates the optimal number of replenishments, m_0, given different values for the parameters W_1 and W_2. More specifically, the parameter W_1 was assumed to take the values 250, 500, 750, ..., 20 000 while W_2 was assumed to take the values 1, 2, 3, ..., 20. The optimal value of m_0 was found by using (14) and observing that the quantities $\Delta\beta_i$, β_i, and α_i are independent of any other parameters. Figure 1 presents m_0, the optimal number of replenishments to be made during period H, as a function of W_1 and W_2.

Relations (14) and (15) also reveal that for a given value of the optimal cost c_0 the value of the parameter W_1 (or W_2) can be determined if the value of the parameter W_2 (or W_1) is given. A computer program was written that considers optimal cost with values 100, 200, 300, ..., 2 000 and calculates W_2 for a given W_1 value. The parameter W_1 was assumed to take the values 250, 500, 750, ..., 20 000. The resulted observations are plotted in Figures 2 and 3. Figure 2 presents the optimal cost as a function of W_1 and W_2, and Figure 3 presents W_2 as a function of W_1 and the optimal cost.

3. Evaluation of the results

Figure 1 illustrates that the same optimal number of replenishments m_0 can be derived from different combinations of W_1 and W_2 values. In particular, common values of m_0 form the regions in Figure 1, where the borders fit straight lines. Linear regression analyses of the observations laying on the border lines reveal that the above observations perfectly fit straight lines.

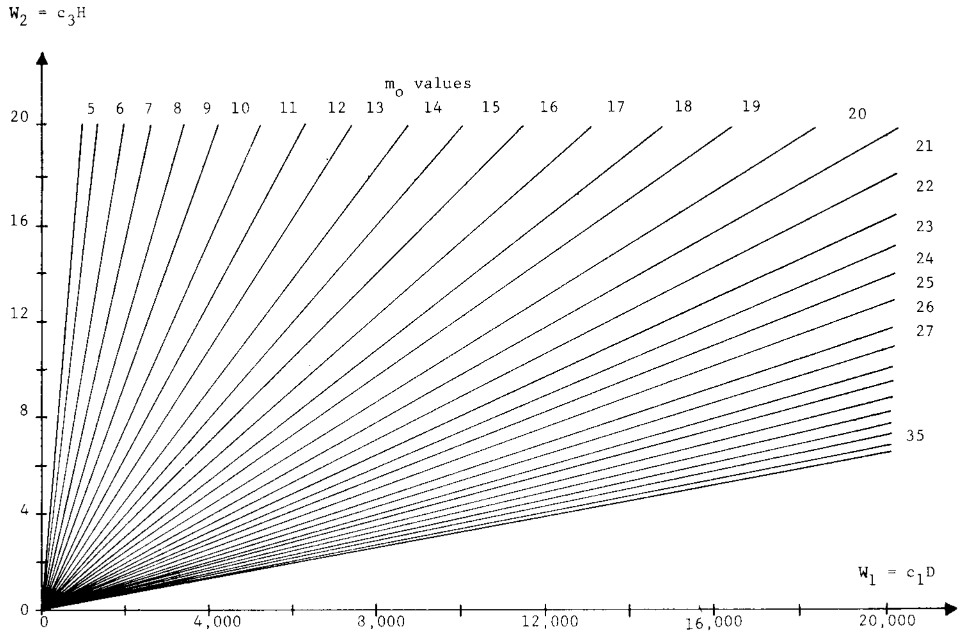

Fig. 1. Optimal number of replenishments, m_0, versus demand and planning horizon

An examination of Figure 1 suggests that the optimal number of replenishments m_0 is highly sensitive (narrow regions) when W_1 is large and W_2 is small. The same is also true when W_1 is small and W_2 is large. If we assume that the costs c_1 and c_2 are fixed, then we conclude from the above and the two relations: $W_1 = c_1 D$ and $W_2 = c_2/H$ that the optimal number of replenishments is highly sensitive when the demand is high and the horizon is wide or the demand is low and the horizon is narrow.

However, the optimal number of replenishments m_0 is not so sensitive (wider regions) when both W_1 and W_2 take medium size values. Similarly with the above, if we assume that the unit costs c_1 and c_2 are fixed, then the optimal number of replenishments is not sensitive to the changes of demand D or horizon H when both D and H take medium size values.

The same figure also depicts that the optimal number of replenishments becomes very sensitive when W_1 and W_2 take small values. The opposite is true when the values of both W_1 and W_2 are large. That is, when the c_1 and c_2 are fixed, then the optimal number of replenishments is highly sensitive when the demand is low and the horizon is wide. However, the number of optimal replenishments is not so sensitive when the demand is high and the horizon is narrow.

Figure 2 illustrates that the same value of optimal cost c_0 can be achieved from different combinations of W_1 and W_2 values. If it is assumed that c_1 and c_2 are constant, then the optimal cost c_0 is fixed if by increasing the demand (or equivalently, the parameter W_1) the horizon increases (or equivalently, the parameter W_2 decreases). Similarly, if the demand takes small values (how small depends on the optimal cost level), then in order for the optimal cost to be constant the horizon should be decreased (i.e., the parameter W_2 be increased). Moreover, if the optimal cost is small, so is the threshold before which high changes of the horizon occur. Since the iso-cost curves become closer for high values of demand, we can conclude that the optimal cost becomes highly sensitive when the demand takes large values. The above behavior of the optimal cost is more dramatic when the cost takes small values (then the iso-cost curves tend to be linear sooner for small cost values).

Figure 3 depicts the relation between W_1 and the optimal cost when W_2 is fixed. If c_1 and c_2 are constants, then Figure 3 suggests that when the demand is small the optimal cost changes dramatically even with small changes in the demand. The plots in Figure 3 tend to become linear for high values of W_1. This behavior indicates that if c_1 and c_2 are constant, then for a constant horizon the optimal cost

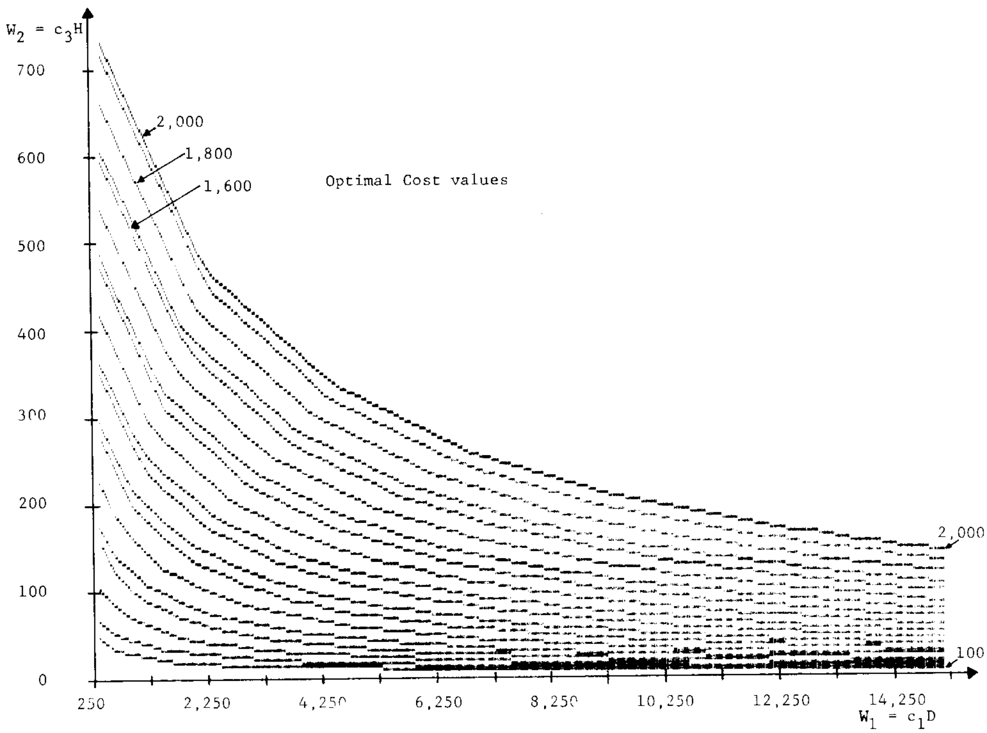

Fig. 2. Optimal cost values versus demand and planning horizon

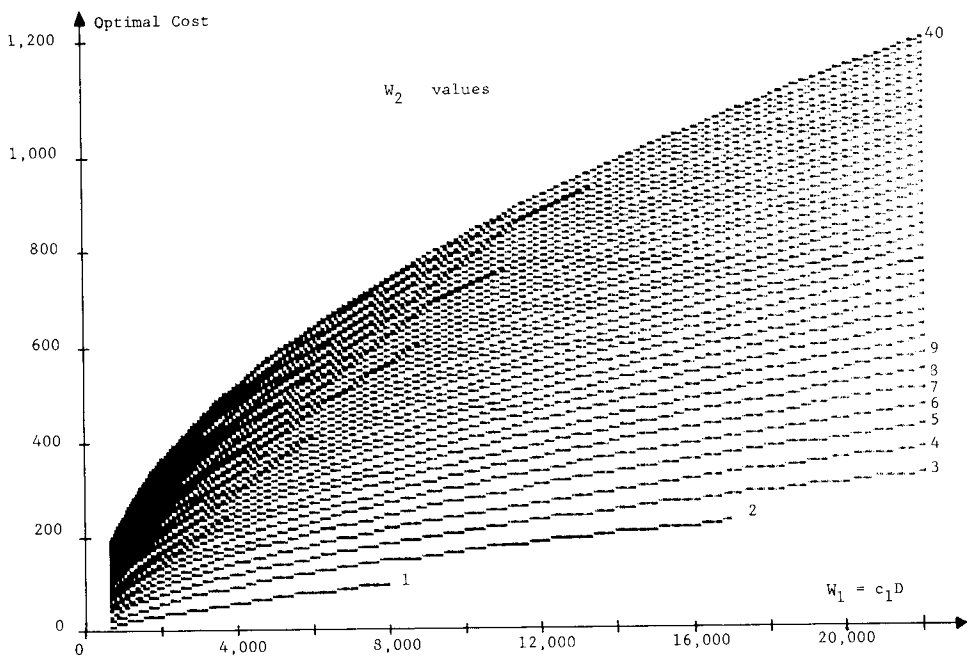

Fig. 3. Optimal horizon versus optimal cost and demand

depends linearly on the demand when the demand is high. Again, how much high is considered high depends on the current horizon length. The wider the horizon is (i.e., the smaller the W_2 parameter becomes), the more rapidly the relation between optimal cost and demand becomes linear.

4. Concluding remarks

Very often in real life situations the parameters are not constant or cannot be determined with accuracy. As this paper illustrates some of the parameters in the (t_i, S_i) inventory system play a critical role under certain conditions. A deep understanding of the sensitivity of the (t_i, S_i) model under various conditions makes it to be even more applicable to real life problems.

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References

[1] W.A. Donaldson, "Inventory replenishment policy for a linear trend in demand – an analytical solution", *J. Oper. Res. Soc.* **28** (1977) 663–670.
[2] R.J. Henery, "Inventory replenishment policy for increasing demand", *J. Oper. Res. Soc.* **30** (1979) 611–617.
[3] A. Mitra, J.F. Cox and R.R. Jesse, Jr., "A note on determining order quantities with a linear trend in demand", *J. Oper. Res. Soc.* **35** (1984) 141–144.
[4] E. Naddor, *Inventory Systems*, Wiley, New York, 1984.
[5] R.I. Phelps, "Optimal inventory rule for a linear trend in demand with a constant replenishment period", *J. Oper. Res. Soc.* **31** (1980) 439–442.
[6] E. Ritchie, "Practical inventory replenishment policies for a linear demand in demand followed by a period of steady demand", *J. Oper. Res. Soc.* **31** (1980) 605–613.
[7] E. Ritchie, "The E.O.Q. for linear increasing demand: A simple optimal solution", *J. Oper. Res. Soc.* **35** (1984) 949–952.
[8] E.A. Silver, "A simple inventory replenishment decision rule for a linear trend in demand", *J. Oper. Res. Soc.* **30** (1979) 71–75.