Atomic Routing Games on Maximum Congestion

Costas Busch, **Malik Magdon-Ismail**
{buschc,magdon}@cs.rpi.edu

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Outline

- Motivation and Problem Set Up;

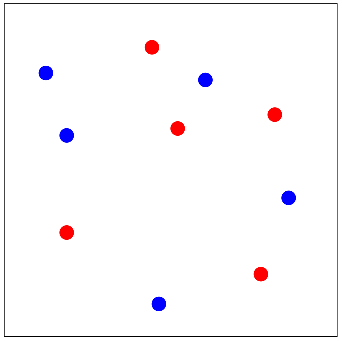
- Related Work and Our Contributions;

- Proof Sketches;

- Wrap Up.

Routing

Routing: coustruct “good” paths given sources and destinations.



- Communication Networks – eg. **Internet**.
- Ad-hoc Networks – eg. sensor networks.
- Parallel Architectures – eg. Mesh.
- . . .

Routing

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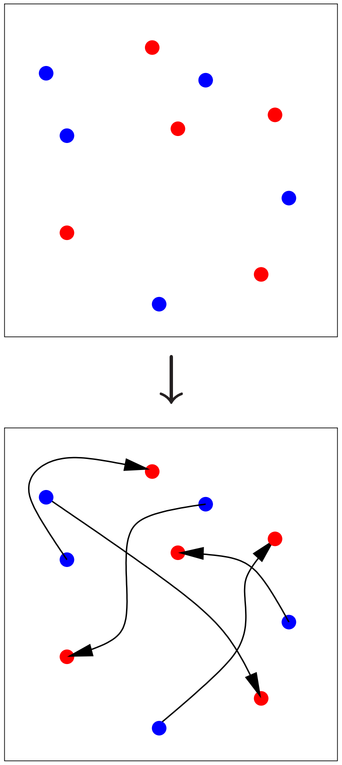

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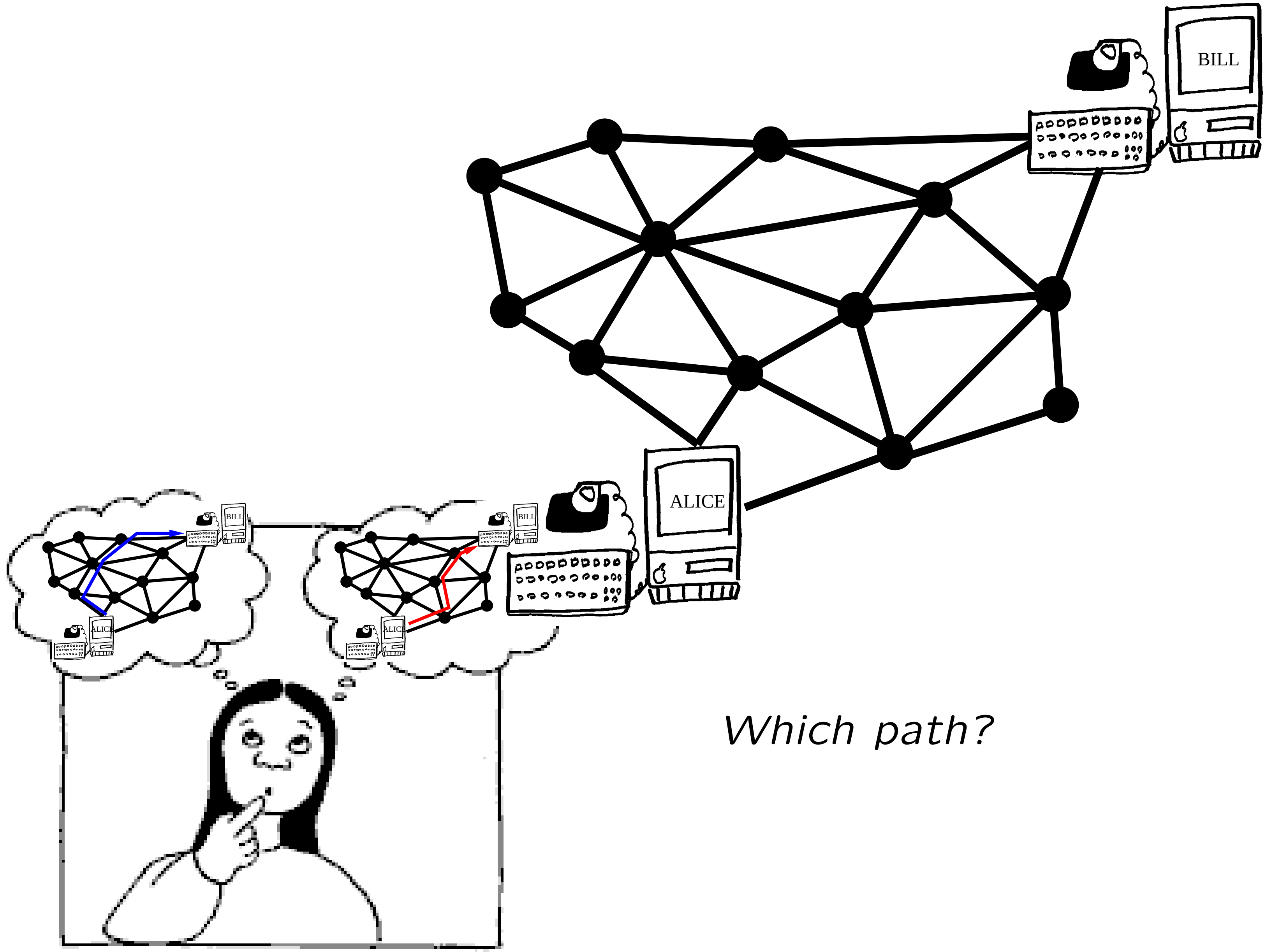
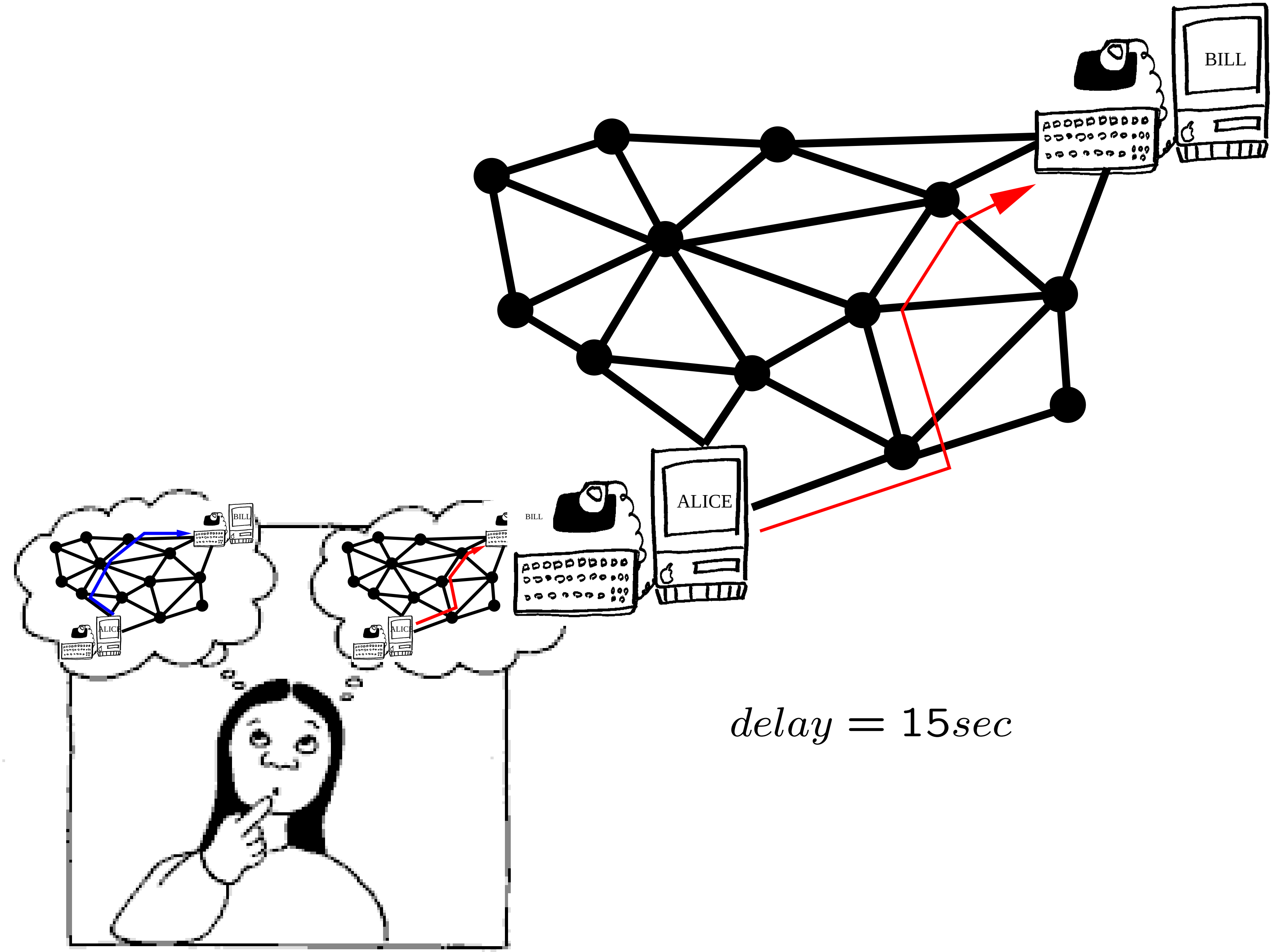
Motivation



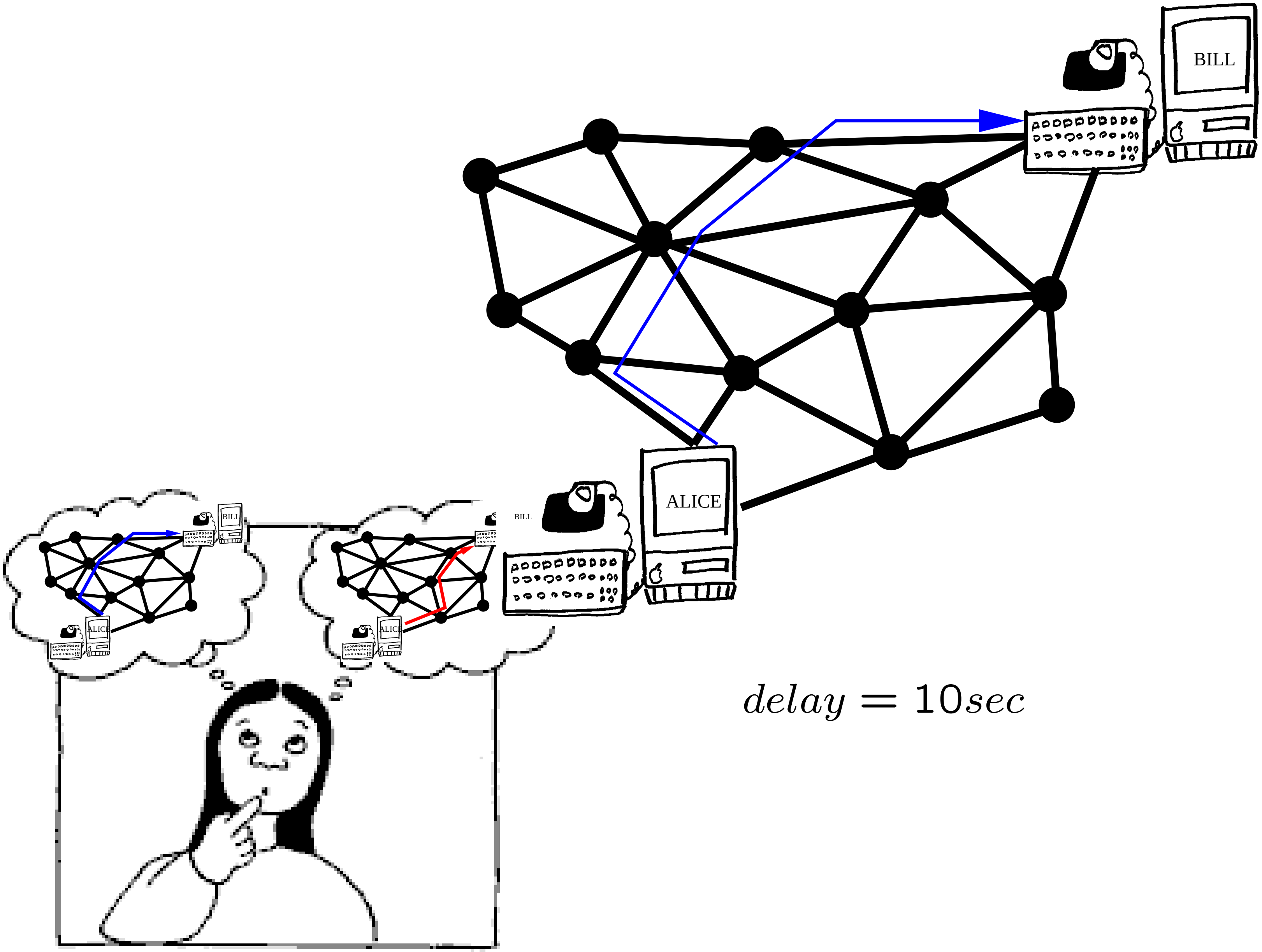
Which path?

Motivation



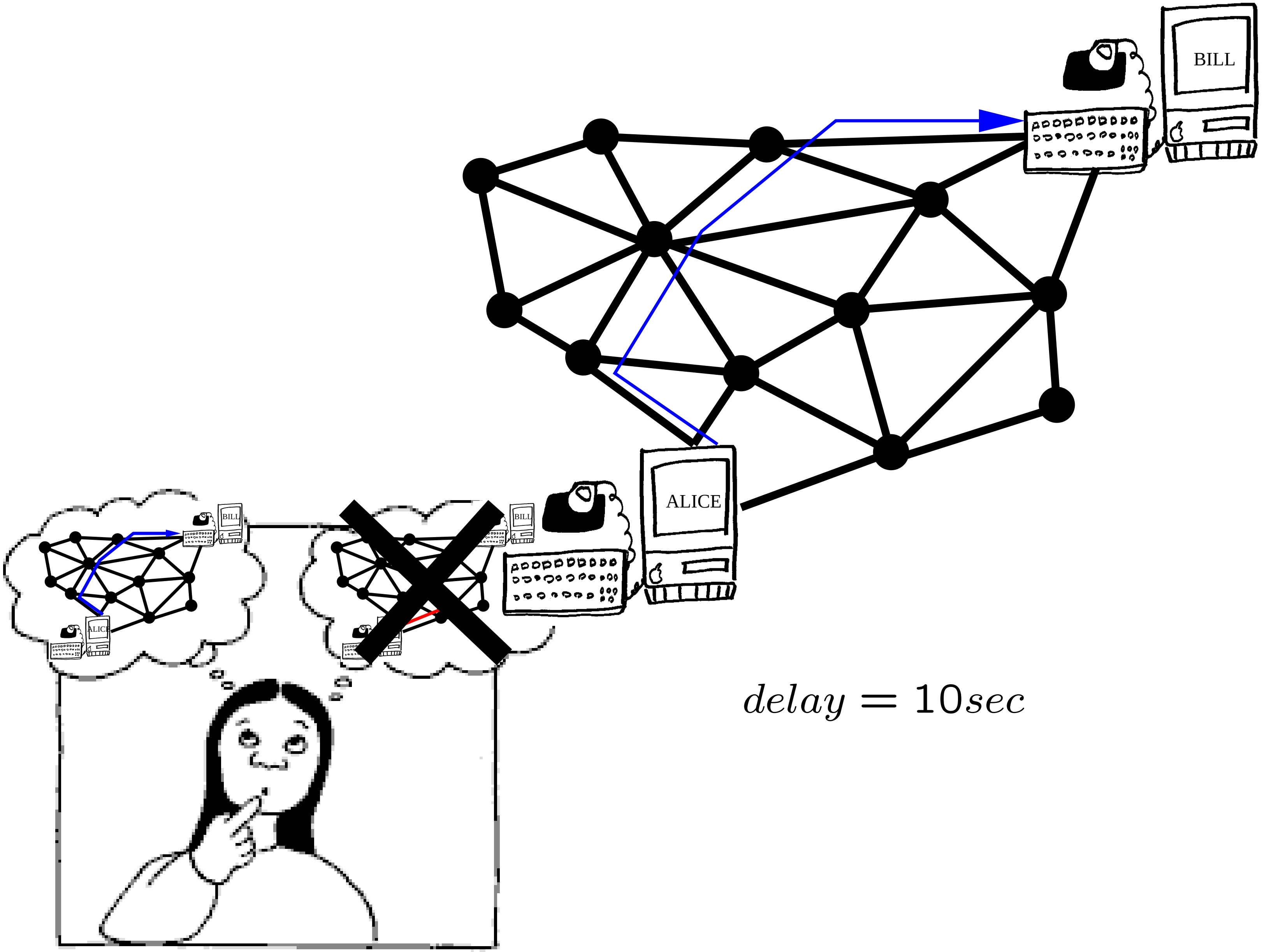
$$delay = 15sec$$

Motivation



$delay = 10sec$

Motivation



$delay = 10sec$

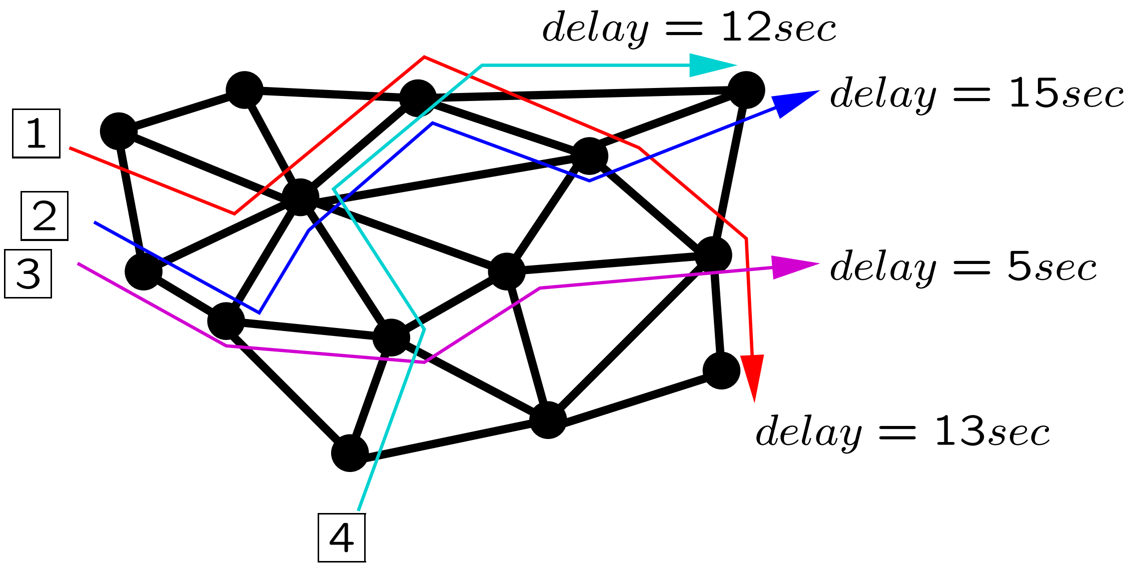
Routing Games

- **Selfish players**: everyone will change paths to minimize their delay.
  Best Response Dynamic

- **Nash-Routing**: no-one wishes to change her path selection, given what everyone else is doing.

We study properties of this process.

Routing Games



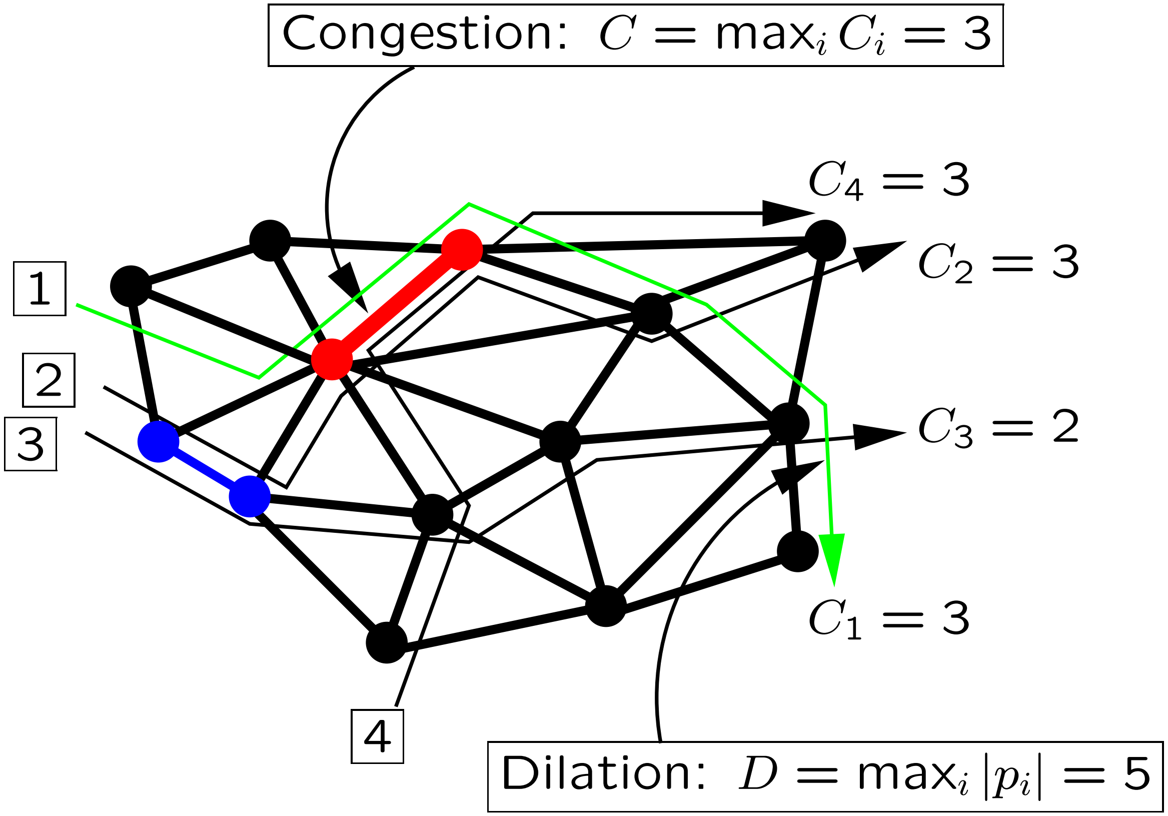
Player cost pc_i: delay of player i's packet.

Social cost SC: maximum delay over all players.

$$\boxed{SC = 15sec}$$

Players minimize their player cost selfishly
Ideally, social cost should be minimized.

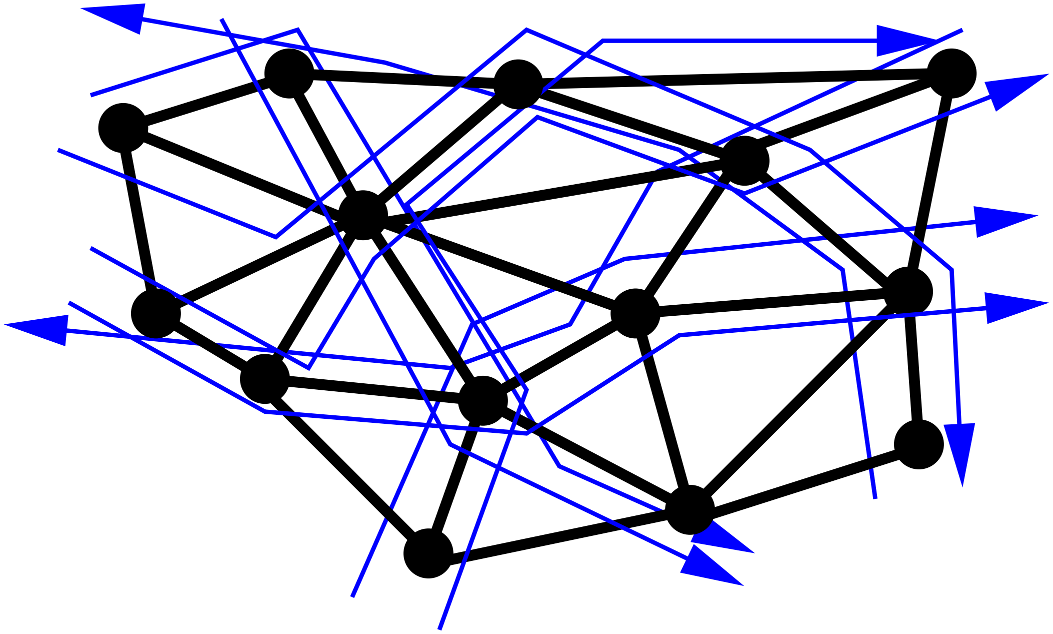
Quantifying Delay



C_i is the largest congestion on path p_i.

Social Cost: $\max_i delay_i = O(C + D)$ [LMR95]
Player Cost: $delay_i = \tilde{O}(C_i + |p_i|)$ [BS95]

Congested Networks



$$C \gg D, C_i \gg |p_i|$$

Social Cost: $\max_i delay_i = O(C)$

Player Cost: $delay_i = \tilde{O}(C_i)$

Formal Setup

Routing (Congestion) Game: $(\mathbf{N}, G, \{\mathcal{P}_i\}_{i \in \mathbf{N}})$.

$\mathbf{N} = \{1, 2, \ldots, N\}$ – players, i.e. $(source, dest)$ pairs;
$G = (V, E)$ – network;
$\mathcal{P}_i$ – strategy sets (*edge-simple* paths).

Routing: $\mathbf{p} = [p_1, p_2, \cdots, p_N]$ – *pure strategy profile*.

Congestion: $C_e(\mathbf{p}) = \#$ paths using edge e.

Path Congestion: $C_i(\mathbf{p}) = \max_{e \in p_i} C_e(\mathbf{p})$;
Network Congestion: $C(\mathbf{p}) = \max_i C_i(\mathbf{p})$;

Social Cost: $SC(\mathbf{p}) = C(\mathbf{p})$ (Network Congestion).

Player Cost: $pc_i(\mathbf{p}) = C_i(\mathbf{p})$ (Player's Path Congestion).

Nash-routing $\mathbf{p}$: $pc_i(\mathbf{p}) \leq pc_i(\mathbf{p}')$ ($\mathbf{p}'$ differs from $\mathbf{p}$ only in p_i).

(No one can unilaterally inprove her situation in a Nash-routing.)

Quality of Nash-Routings

Prics of Stability $PoS = \inf_{\mathbf{p}\in\ \mathbf{P}} \frac{SC(\mathbf{p})}{SC*},$

Price of Anarchy $PoA = \sup_{\mathbf{p}\in\ \mathbf{P}} \frac{SC(\mathbf{p})}{SC*}.$

PoS: minimum price for stability. (best possible selfish outcome)
PoA: maximum price for stability. (worst possible selfish outcome)

Ideal: $PoS = PoA = 1$.

Related Work

	Atomic Flow	Splittable Flow
Pure	⬛, 🟥, **[BM06]**	⬛
Mixed	🟥	⬛, 🟥

	Max SC	Sum SC	Other SC	
Max pc	**[BM06]**	–	–	🟦
Sum pc	⬛, 🟥	⬛, ⬛	🟥	🟦

🟥: specific network or strategy sets (eg. parallel links or singleton sets).

🟦: existence or convergence to equilibrium (do not look at quality (SC)).

Note: sum SC is relevent when network resources, not max. player delay is important.

Our Contribution – PoS

Routing games with max. player/social costs on general networks.

Theorem 1

(i) $PoS = 1$;

(ii) All best response dynamics converge to a Nash-routing

$$SC(\mathbf{p}_{final}) \leq SC(\mathbf{p}_{start}).$$

– There exist good Nash-routing.

– Starting at any good routing, selfish players can only improve!

Good oblivious starting routings: [MMVW97], [R02], [BMX05].

Our Contribution – PoA

Routing games with max. player/social costs on general networks.

Theorem 2 $PoA < 2(\ell + \log n)$.

ℓ upper bounds path lengths in the strategy sets.

ℓ can be small (eg. Hypercubes).

Theorem 3 $\kappa_e - 1 \leq PoA \leq c(\kappa_e^2 + \log^2 n)$.

$\kappa_e(G)$ is the length of the longest cycle.

PoA is bounded by topological properties of the network.

Proof Sketch: $PoS = 1$

Establish a total order $\leq_c, <_c$ among routings with:

Lemma 1 There exists a minimum routing $\mathbf{p}^*$. [Compactness of routings.]

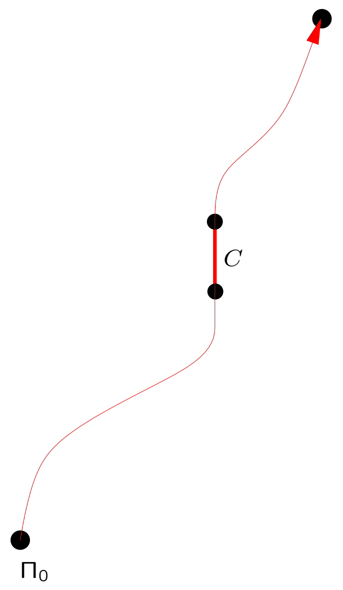
Lemma 2 $SC(\mathbf{p}) \leq SC(\mathbf{p}')$ iff $\mathbf{p} \leq_c \mathbf{p}'$.

Lemma 3 If $\mathbf{p} \rightarrow \mathbf{p}'$ in a selfish move, then $\mathbf{p}' <_c \mathbf{p} \implies SC(\mathbf{p}') < SC(\mathbf{p})$.

Corollary Minimum routings $\mathbf{p}^*$ are a Nash-routings. Best response dynamics converge to better Nash-routing.

(Note: cf. potential function methods.)

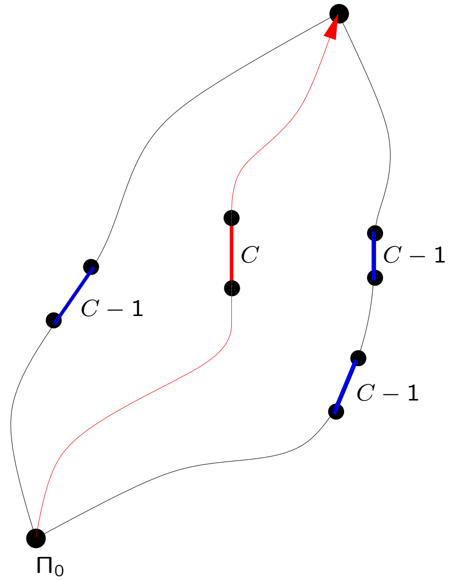
Proof Sketch: $PoA \leq 2(\ell + \log n)$



E_0: Edges of congestion C.

Π_0: Players using edges in E_0.

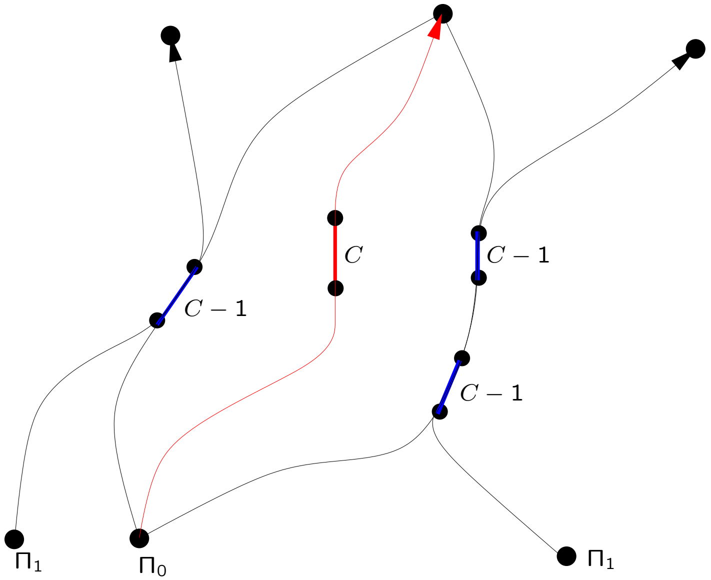
Proof Sketch: $PoA \leq 2(\ell + \log n)$



Alternative paths for players in Π_0 must all have at least one edge with congestion at least $C-1$.

$$\left(\begin{array}{l} E_0\text{: Edges of congestion } C. \\ \Pi_0\text{: Players using edges in } E_0. \end{array} \right)$$

Proof Sketch: $PoA \leq 2(\ell + \log n)$



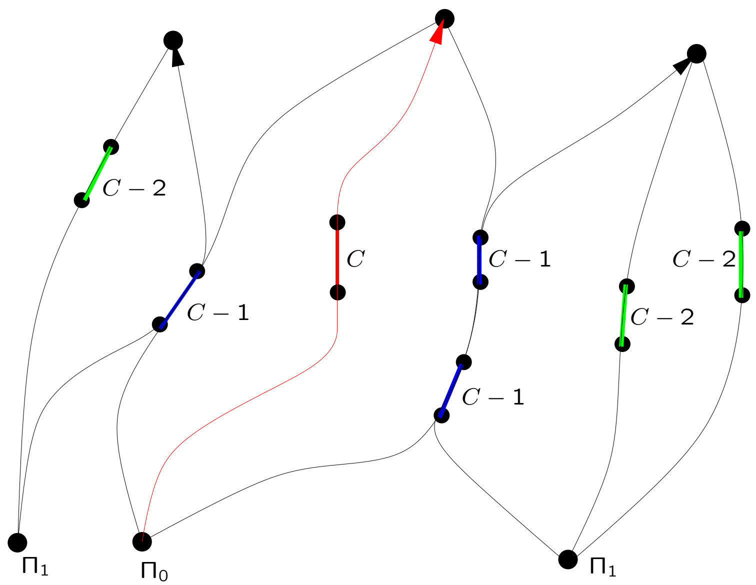
E_1: All these edges of congestion $\geq C - 1$.
Π_1: Players using edges in E_1.

if $|E_1| \leq 2|E_0|$, stop, else continue **Edge Expansion Process**

$(|E_0| = 1, |E_1| = 4)$

(E_1 is formed from all possible paths of players in Π_0)

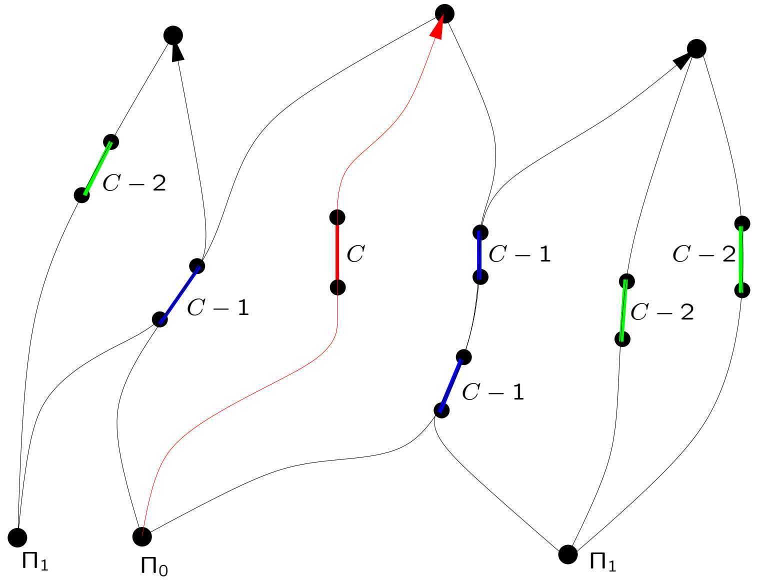
Proof Sketch: $PoA \leq 2(\ell + \log n)$



Alternative paths for players in Π_1 must all have at least one edge with congestion at least $C-2$.

$$\left(\begin{array}{l} E_1\text{: Edges of congestion at least } C-1. \\ \Pi_1\text{: Players using edges in } E_1. \end{array} \right)$$

Proof Sketch: $PoA \leq 2(\ell + \log n)$



E_2: All these edges of congestion $\geq C-2$.

if $|E_2| \leq 2|E_1|$, stop.

$(|E_1| = 4, |E_2| = 7)$

(E_2 is formed from all possible paths of players in Π_1)

Proof Sketch: $PoA \leq 2(\ell + \log n)$

$E_0 \quad E_1 \quad \ldots \quad E_{s-1} \quad E_s$

$\Pi_0 \quad \Pi_1 \quad \ldots \quad \Pi_{s-1}$

$\boxed{s \leq \log n}$

(Each step doubles the size of E_i.)

Max. # times edges used by packets in Π_{s-1}

Min. # times edges in E_{s-1} used (only packets in Π_{s-1} use edges in E_{s-1})

$$|\Pi_{s-1}| \cdot \ell \geq (C - (s-1)) \cdot |E_{s-1}|$$

$$C_{opt} \geq \frac{|\Pi_{s-1}|}{|E_s|} \geq \frac{|\Pi_{s-1}|}{2|E_{s-1}|}$$

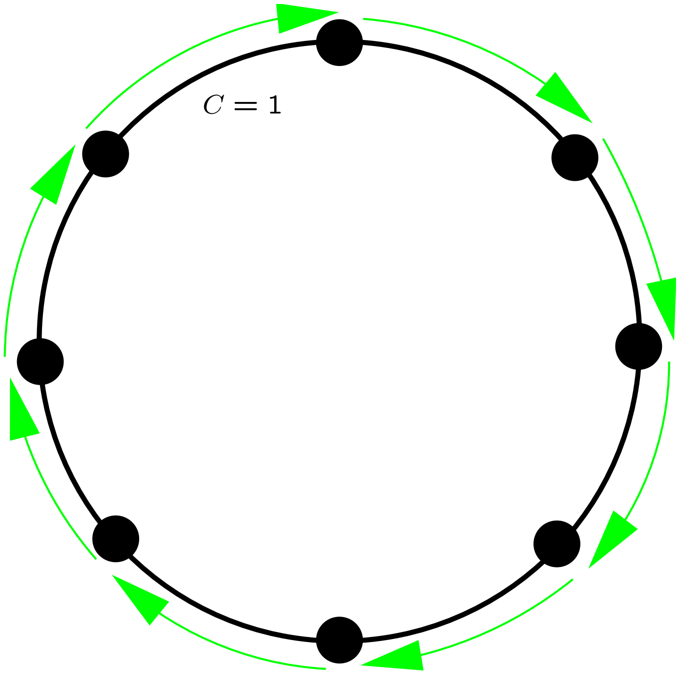
Optimal C

Every packet in Π_{s-1} must use at least one edge in E_s

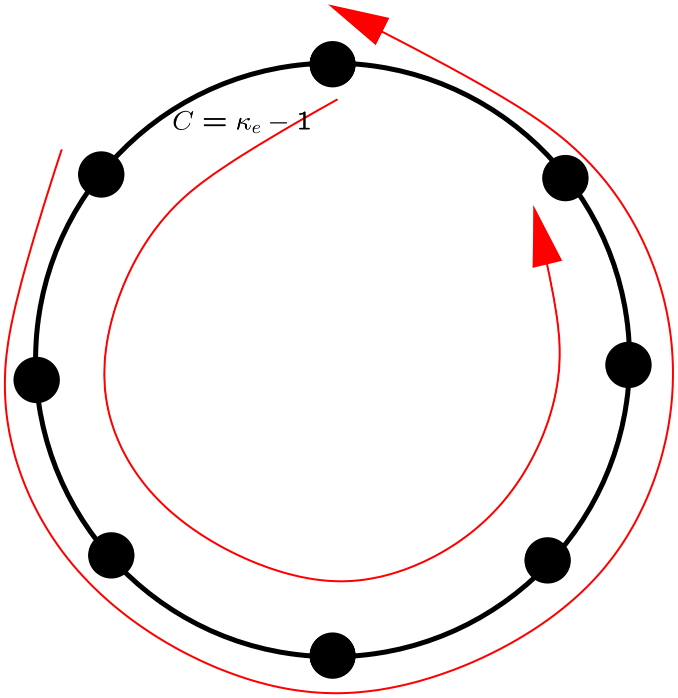
$|E_s| \leq 2|E_{s-1}|$

$$PoA = \frac{C}{C_{opt}} \leq 2\ell + s - 1.$$

Proof Sketch: $\kappa_e - 1 \leq PoA \leq c(\kappa_e^2 + \log n)$



Optimal Nash-routing
(Players use shortest paths)
$C = 1$



Worst Case Nash-routing
(Players use longest paths)
$C = n - 1 = \kappa_e - 1$

If network is not a cycle, use the largest cycle in the network.

Proof Sketch: $\kappa_e - 1 \leq PoA \leq c(\kappa_e^2 + \log n)$

Combinatorial Lemma If G is 2-connected, then $\kappa_e(G) \geq \sqrt{2\ell} - \frac{3}{2}$.

2-connected Networks:

$\ell = O(\kappa_e^2)$, so

$PoA \leq 2(\ell + \log n) \implies PoA = O(\kappa_e^2 + \log^2 n)$.

General Networks:

Step 1: Decompose G: tree of 2-conected and acyclic components.

Step 2: Many players satisfied in some 2-connected component;

Step 3: Extend $PoA \leq 2(\ell + \log n)$ to **Partial Nash-routing**.

Step 4: Use 2-connected and Partial Nash-routing results.

Wrap Up

- Studied general congestion games with **max. social/player costs**.
- Appropriate metrics when delays are important in congested networks.
- $PoS = 1$ and selfish dynamics are good.
- **Path Length Bound on** PoA: $PoA \leq 2(\ell + \log n)$.
- **Topological bounds on** PoA: $\kappa_e - 1 \leq PoA \leq c(\kappa_e^2 + \log^2 n)$.
- **Conjecture[Lower bound is tight]**: $PoA \leq \kappa_e$.
- Non-congested networks: $SC = C + D$; $pc_i = C_i + |p_i|$?

Thank You!

http://www.cs.rpi.edu/~magdon