A Note on Online Steiner Tree Problems

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CCCG 2014
Steiner tree problem

Given an arbitrary undirected graph $G$ with weights (lengths) on edges

- terminals
- Steiner or optional vertices

Steiner tree $T$ for terminals is a subgraph connecting all of them minimizing $T$’s length
Steiner tree problem (STP)

Steiner trees have applications in

- VLSI design
- Computational biology
- Network and group communication

Classic Steiner Tree Problem (STP)

- All the terminals are known in advance
- NP-hard problem
- Approximation solutions are given comparing length of with the length of the optimal tree $\text{OPT}$
- Current best approximation $\ln(4) + \varepsilon < 1.39$
Online Steiner Tree Problem (OSTP)

- Terminals appear one at a time online; future terminals are not known.

- $T$ is formed step by step.

- Lower bound of $\frac{1}{2} \log k$.

- Current best approximation of $\log k$.

$k$ is the number of terminals.
Graph

Steiner tree

Cost 3
Terminals arrive in some order
Online Steiner tree

Graph

Terminals arrive in some order
Graph

Online Steiner tree

Terminals arrive in some order
Online Steiner tree

Terminals arrive in some order
Graph

Online Steiner tree

Terminals arrive in some order
Online Steiner tree

Graph

Terminals arrive in some order
Online Steiner tree

Terminals arrive in some order
Graph

Online Steiner tree

$\text{Cost} = 6 - \varepsilon$
Optimal Steiner tree

Cost 3

Online Steiner tree

Cost = $6 - \varepsilon \approx 2 \cdot \text{OPT}$
New Problem: Bursty Arrival of Terminals

Bursty Steiner tree problem (BSTP)
- **Instance**: Graph $G$ and a set $S$ of terminals appearing online in a sequence of $m$ groups (bursts), $m \leq k$
- **Question**: Find a Steiner tree for $S$ in $G$ with minimum length

Captures known problems

$m = 1$ \hspace{1cm} Classic STP

$m = k$ \hspace{1cm} Online STP
Terminals arrive in groups
Terminals arrive in groups
Terminals arrive in groups
Graph

Bursty Steiner tree

Cost = 5
Optimal Steiner tree

Online Steiner tree

Bursty Steiner tree

Cost = 3

Cost = 6 - \varepsilon

Cost = 5
Contributions

Bursty STP:
Tight approximation bound of

$$\Theta\left(\min\{\log k, m\}\right)$$

$k$ is the number of terminals

$m$ is the number of bursts

Online STP:
$$\Theta\left(\log k\right)$$
Terminal Steiner tree problem (TSTP)

- All the terminals are the leaves of the Steiner tree $T$
- TSTP is also NP-Hard like STP with best approximation 2.17

Bursty TSTP contributions:
- Lower bound of $\min\left\{ \frac{\log k}{4}, \frac{m}{4} \right\}$
- Upper bound of $\min\{2\log k, 3\lambda m\}$

$\lambda < 2.17$ is the current best approximation of TSTP
BSTP in directed graphs

- Networks may contain links that are asymmetric in QoS they offer

- Asymmetry \( \alpha = \max_{\{u,v\} \in G} \frac{w(u,v)}{w(v,u)} \)

  \( w(u,v) \) is the weight of the edge \((u,v)\)

- We prove near-tight bounds for bounded \( \alpha \)

\[
O\left(\min\left\{\max\left\{\alpha \frac{\log k}{\log \alpha}, \alpha \frac{\log k}{\log \log k}\right\}, k, \alpha m\right\}\right)
\]

\[
\Omega\left(\min\left\{\max\left\{\alpha \frac{\log k}{\log \alpha}, \alpha \frac{\log k}{\log \log k}\right\}, k^{1-\varepsilon}, \max\left\{\alpha, \alpha \frac{m}{\log \alpha}\right\}\right\}, \varepsilon > 0\right)
\]
BSTP in undirected graphs

The upper bound $O(\min\{\log k, m\})$ proof

The lower bound $\Omega(\min\{\log k, m\})$ proof
The upper bound $O(\min\{\log k, m\})$ proof

**Algorithm for O(m):**

when a new burst $B_i$ arrives

1. Compute a Steiner tree $T_i$ for $B_i$

2. Find a $v \in B_i$ closest to existing tree $T$ of previous bursts and connect $v$ to $T$
Optimal Steiner tree within each group
The upper bound \(O\left(\min\{\log k, m\}\right)\) proof

- \(O(\log k)\) is from Online STP

- For \(O(m)\), if we maintain \(m\) Steiner trees, one for each burst \(B_i\), we obtain \(2\phi m\) approximation, \(\phi < 1.39\) is approx. of STP
The upper bound $O(\min\{\log k, m\})$ proof

- The factor of $2$ is because to join individual trees $T_i$ to obtain $T$, we may need a path that has length as most $\text{OPT}$

Q.E.D.
The lower bound $\Omega(\min\{\log k, m\})$ proof

Idea: create a sequence of terminal burst instances based on a sequence of graphs and apply adversarial argument

Variation of OSTP lower bound [Imase and Waxman 91]
The lower bound $\Omega(\min\{\log k, m\})$ proof

Consider a sequence of $m$ bursts $B = \{B_0, ..., B_m\}$ for graphs $G_i, i \geq 0$

There exists a path $p$ between $v_0$ and $v_1$ for all the terminals in $B$ with length exactly 1 in $\text{OPT}$

Minimum tree sequence: $\overline{T} = \{\overline{T}_0, ..., \overline{T}_m\}$ for $G_i, i \geq 0$ w.r.t. $B = \{B_0, ..., B_m\}$ such that $\overline{T}_i$ must contain $\overline{T}_{i-1}$ as a subgraph and connects all the terminals in $B_i$
$G_0$

$STP \ cost = 1$
STP cost = 1
$G_2$

STP cost = 1
$G_3$

**STP cost = 1**
The lower bound $\Omega\left(\min\{\log k, m\}\right)$ proof

- When $|B_i| = 1$ for each $i$, BSTP becomes OSTP, hence $\Omega(\log k)$ applies to BSTP.
- Therefore, we consider the case where $m < \left\lfloor \log k \right\rfloor$ and prove $\Omega(m)$.
- Let $B = \{B_i, \ldots, B_{i+m-1}\}, i \geq 1$ and $B_i$ contains $2^{i-1}$ terminals beside $\nu_0$ and $\nu_1$, $B_{i+1}$ contains $2^i$ terminals, and so on.
The lower bound $\Omega(\min\{\log k, m\})$ proof

- Now for $B_i$ consider $G_i$ and construct $T = \{\bar{T}_i, \ldots, \bar{T}_{i+m-1}\}$

- The length of tree $\bar{T}_i$ is $C_A(\bar{T}_i) \geq 1$ by any algorithm $A$

- $C_A(\bar{T}_{i+1}) \geq 1 + \frac{1}{2}$ as $|B_{i+1}| = 2|B_i|$ and the length of the edges in $G_{i+1}$ are half than $G_i$

- We can achieve this by choosing level $i+1$ nodes that are not in $T_i$
$G_0$ \hspace{1cm} \hspace{1cm} $T_0$

$\text{cost} = 1$

Group 0

$\text{cost} = 1$
$G_1$

$T_1$

$\text{cost} = 1$

$\text{cost} = 1 + \frac{1}{2}$
$G_2$

$\frac{1}{4}$

$\frac{1}{2}$

$1$

$\frac{1}{4}$

$\frac{1}{4}$

$\frac{1}{4}$

$\frac{1}{4}$

$cost = 1$

$T_2$

$\frac{1}{2}$

$\frac{1}{2}$

$1 + \frac{1}{2} + \frac{1}{2}$

Group 2
\[
\begin{align*}
\text{Group 3} \\
\text{cost}^+ & = 1 \\
\text{cost} & = \frac{1}{2} + \frac{1}{2} + \frac{1}{240}
\end{align*}
\]
The lower bound $\Omega(\min\{\log k, m\})$ proof

- The length of the shortest path from each node in $B_{i+j}$ to a node in $\overline{T}_{i+j-1}$ is $\frac{1}{2^{i+j}}$

- Moreover, $2^{i+j-1}$ level $i+j$ nodes are in terminals in $B_{i+j}$

- $C_A(\overline{T}_{i+j}) \geq C_A(\overline{T}_{i+j-1}) + \frac{1}{2} = 1 + \frac{j-1}{2}$

- This is $\Omega(m)$ as there are exactly $m$ groups in $B$  

Q.E.D.
BTSTP in complete graphs

• Results follow the proof structure of BSTP, but in complete graphs

• And graph sequence construction and adversarial argument are more involved

• Bounds are tight with in a constant factor

  Lower bound
  \[ \min\left\{ \frac{\log k}{4}, \frac{m}{4} \right\} \]

  Upper bound
  \[ \min\{2 \log k, 3\lambda m\} \]
Conclusions

Tight and near-tight results for online Steiner tree problem variations

These variations provide trade-offs to existing solutions

Open problems:

• Provide similar trade-offs for other Steiner tree variations, e.g. node-weighted, group Steiner, etc.