Tensor Visualization

Tensor data

- A tensor is a multivariate quantity
  - Scalar is a tensor of rank zero $s = s(x,y,z)$
  - Vector is a tensor of rank one $v = (v_x, v_y, v_z)$
  - For a symmetric tensor of rank 2, its nine components A_{ij} are related by $A_{ij} = A_{ji}$ for $i,j = 1,2,3$.

- A tensor field is a field which associates a tensor with each point in space

- Examples are

  Stress tensor

  Strain tensor

  Momentum flux density tensor

  DT-MRI: Diffusion tensor magnetic resonance imaging

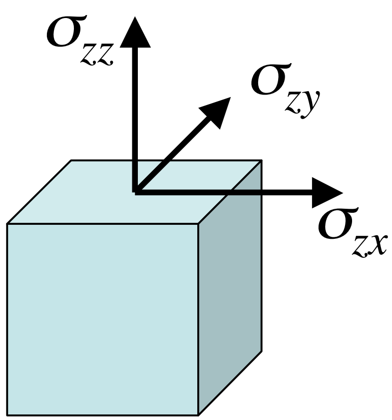
Stress Tensor

- Stress tensor describes the state of stress in a 3D material
- Diagonal components: normal stresses
  - compression or tension
  - Act perpendicular to the surface
- Off-diagonal components: shear stresses
  - act tangentially to the surface

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Symmetric tensor

$$\sigma_{xy} = \sigma_{yx}; \sigma_{yz} = \sigma_{zy}; \sigma_{xz} = \sigma_{zx}$$



Tensor Eigenvalue Equation

- The eigenvectors and eigenvalues of tensor (matrix) A are obtained as follows

$$A \bullet x = \lambda x$$

$$\det | A - \lambda I | = 0$$

  - Eigenvectors form a 3D orthogonal coordinate system; axes are called the principal axes of the tensor (directions of normal stresses)

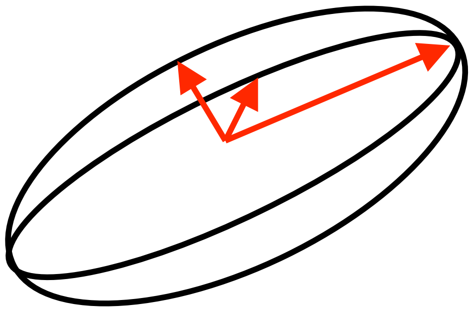
- A 3x3 tensor field $\boldsymbol{A}$ is decomposed into three vector fields called eigenfields (characterized by 3 eigenvectors $\boldsymbol{v}_i$ and 3 eigenvalues $\boldsymbol{\lambda}_i$)

$$v_i = \lambda_i e_i \quad \text{with } i = 1,2,3$$

  - For order $\lambda_1 \geq \lambda_2 \geq \lambda_3$, the vectors v_1, v_2 and v_3 are referred to as the major, medium and minor eigenvectors

Point Icons

- Two types of glyphs used for tensor field visualization
  - Density of the displayed icons must be kept low to avoid visual clutter
- Tensor axes
  - Displaying scaled and oriented principal axes of the stress tensor
- Ellipsoids
  - The principal axes can be taken as minor, medium and major axes of an ellipsoid
  - The shape and orientation of ellipsoid represent the size of the eigenvalues and orientation of the eigenvectors

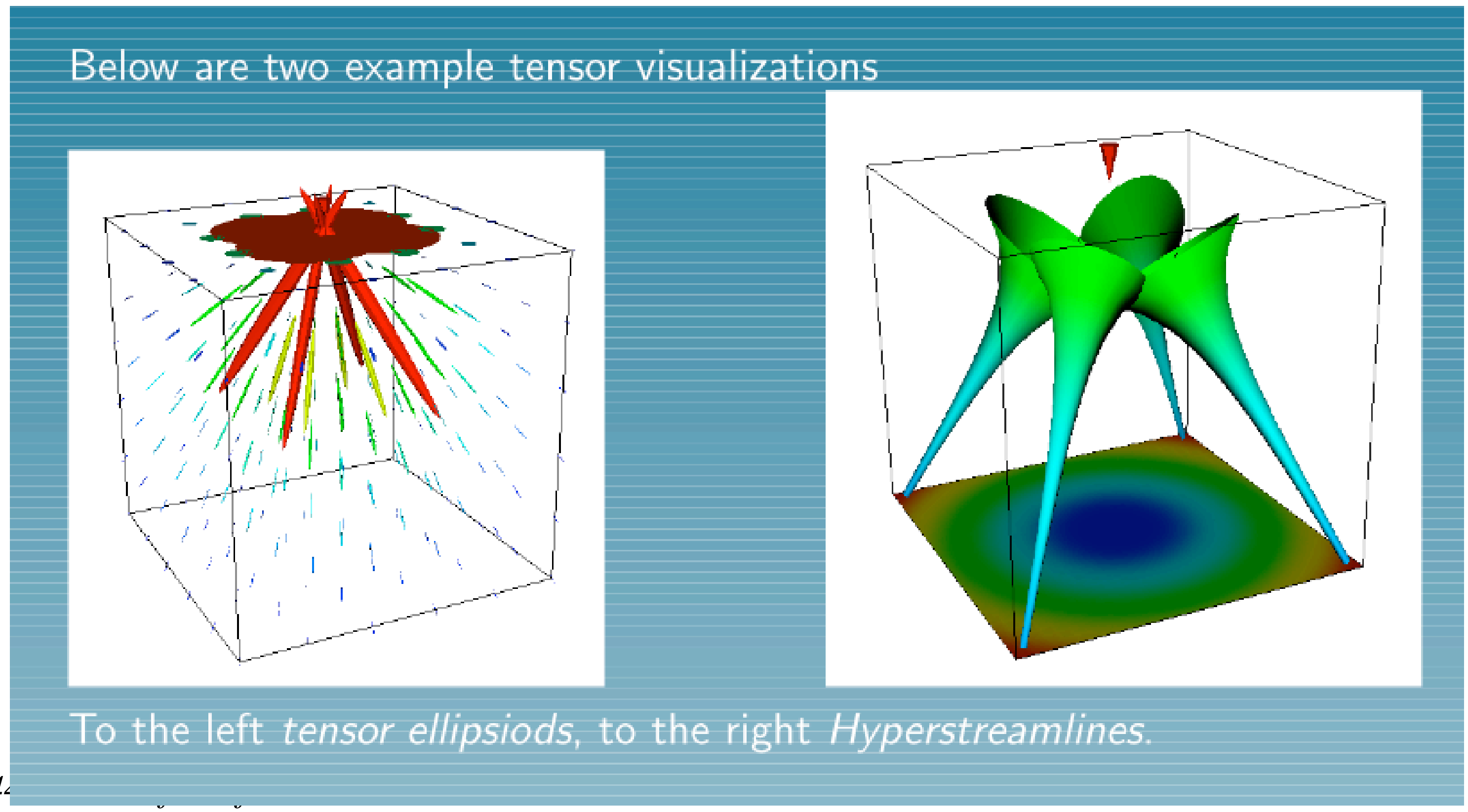


Hyperstreamlines

- An extension of streampolygon technique of a vector field to the case of a tensor field
  - Provide continuous representation of a tensor field
- Constructed by creating a streamline through one of three eigenfields, and sweeping a geometric primitive (ellipse or cross) along the streamline
- Ellipse:
  - Sweeping the ellipse along the eigenfield streamline result in a tubular shape
  - Other two eigenvectors define major and minor axes of the ellipse
- Cross:
  - Sweeping the cross results in a helical shape since the cross arms may rotate in some tensor fields
  - Other two eigenvectors control length and orientation of the cross arms
- Color and trajectory of a hyperstreamline represent the longitudinal eigenvector, and the cross-section encodes two transverse eigenvectors
- Hyperstreamlines can be called major, medium or minor hyperstreamlines depending on the longitudinal eigenvector field.

A Point Load on Surface

- Property of an elastic tensor field produced by a compressive force on the top surface of the material (Boussinesq's problem)
- Analytic expressions for the stress components are known
- Visualizing these analytical results



Global Visualization

- Global visualization is done by encoding the behavior of a large number of hyperstreamlines with display of critical points

- Locus is the set of the critical points in the trajectory of the hyperstreamlines where the longitudinal eigenvector vanishes

- Surface is the locus of points where the cross-section is singular (i.e, reduced to a straight line or a point)

DT-MRI Data

- Characteristic microstructure of the brain's neural tissue, which contains the diffusion of water molecules
  - Anisotropic diffusion:

    diffusivity is greater in some directions than in others.
  - grey matter: largely isotropic
  - white matter: more anisotropic because of the alignment of myelinated neural axons

    It is possible to image the neural pathways connecting the brain

- Fibrous muscle tissue of the heart.

- Diffusion tensor:

$$D = \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix}$$

DT-MRI Visualization

- Combination of scalar, vector and tensor methods

- Scalar metrics:
  - Reducing DT-MRI data to scalar data

    Trace of the diffusion tensor = $D_{xx} + D_{yy} + D_{zz}$

    Ratio = D_{xx} / D_{zz}
  - Combinations of eigenvalues:

    A set of three metrics that measure linear, planar and spherical diffusions

$$C_L = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}; C_P = \frac{2(\lambda_2 - \lambda_3)}{\lambda_1 + \lambda_2 + \lambda_3}; C_S = \frac{3\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}$$

Parameterize a barycentric space in which the three shape extremes (linear, planar, and spherical) are at the corners of a triangle.

More on DT-MRI Visualization

- Eigenvector color maps:
  - Display the spatial patterns of the principal eigenvector only

    the principal eigenvector is aligned with the coherent fibers.
  - R,G,B color according to the X, Y, Z components of the vector.
  - Modulates the saturation of the RGB color with an anisotropic metric.

- Glyphs:
  - Ellipsoids
  - Superquadrics: cylinders for linear ansiotropy, a sphere for isotropy and boxes are for intermediate anisotropies.

- Tractography:
  - To obtain curves of neural pathways

    extraction of the underlying continuous anatomical structures
  - Streamlines, streamtubes, streamsurfaces.