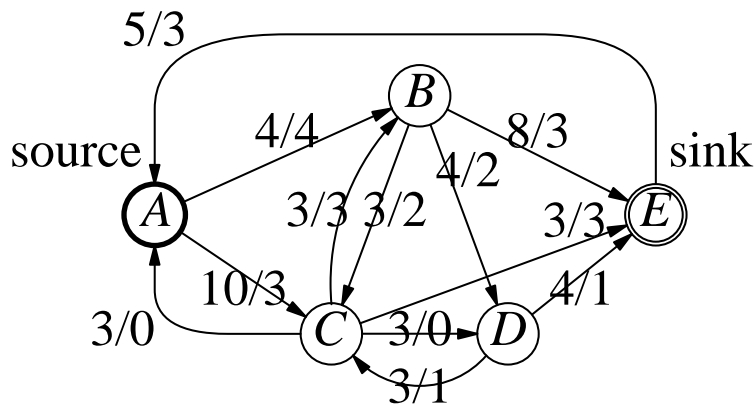


NETWORK FLOW



Four flow-paths:

- 2 along $\langle A, B, C, E \rangle$,
- 1 along $\langle A, B, D, E \rangle$,
- 1 along $\langle A, B, D, C, E \rangle$,
- 3 along $\langle A, C, B, E \rangle$

A flow for $A = \text{source} = s$ and $E = \text{sink} = d$, with net-flow = 4.

Capacitated Network: $\text{capacity } c(x, y) \geq 0$ for each link (x, y) .

Flow and Flow Equation:

- For each link (x, y) , we have an associated *flow* $0 \leq f(x, y) \leq c(x, y)$ with the property that for each node $x \neq \text{source}$ and sink , we have $f^-(x) = f^+(x)$, where

$$f^-(x) = \sum_{y \in N^-(x)} f(y, x) = \text{the total in-flow to } x \text{ and}$$

$$f^+(x) = \sum_{y \in N^+(x)} f(x, y) = \text{the total out-flow to } x.$$

For example, $f^-(B) = 4+3 = 2+2+3 = f^+(B)$.

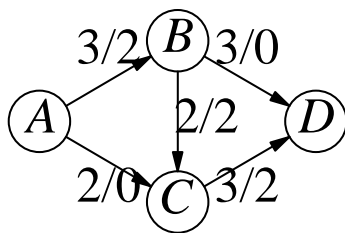
- $f^+(s) - f^-(s) = f^-(d) - f^+(d) = \text{net-flow from } s \text{ to } d$.
The net-flow increases from 4 to 7 if we let $f(E, A) = 0$.
- Net-flow does not change if we eliminate flow around a cycle. (The net-flow is not affected if we reduce the flow around $\langle B, D, C, B \rangle$ by $f(B, D) \wedge f(D, C) \wedge f(C, B) = 1$.)
- For maximizing net-flow, $f(d, s) = 0$.

Question: How to find flow-paths from a given flow?

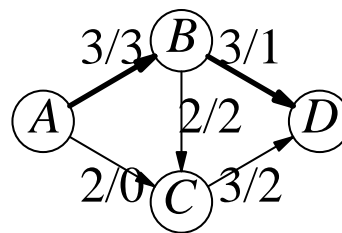
FLOW-AUGMENTATION

Simple Flow-augmentation (does not reduce existing $f(x, y)$):

- (1) Find a path $\pi = \pi(\text{source}, \text{sink})$ each of whose links (x, y) is underused, i.e., $f(x, y) < c(x, y)$.
- (2) For each $(x, y) \in \pi$, increase $f(x, y)$ by $m = \min \{c(x, y) - f(x, y) : (x, y) \in \pi\} > 0$. Net-fbw increased = m .

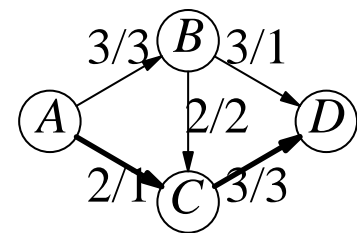


A fbw of 2.



Aug. path

$\langle A, B, D \rangle: 1$

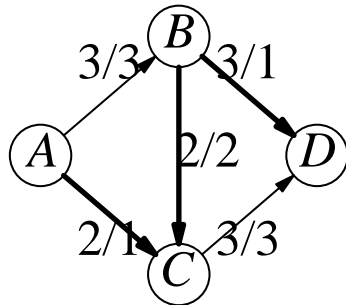


Aug. path

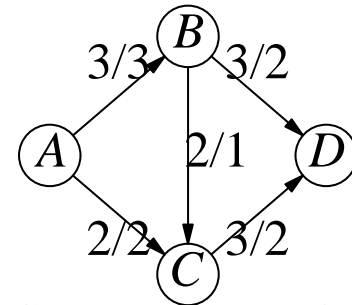
$\langle A, C, D \rangle: 1$

Complex Flow-augmentation (reduces some existing $f(x, y)$):

- (1) Find a path $\pi = \pi(\text{source}, \text{sink})$ with the properties:
 - (a) Each link (x, y) used in forward-direction in π is under-used, i.e., $f(x, y) < c(x, y)$.
 - (b) Each link (x, y) used in backward-direction in π has a positive fbw, i.e., $f(y, x) > 0$.
- (2) Update the link-fbws as follows:
 - (a) Let $m = m^+ \wedge m^-$, where $m^+ = \min \{c(x, y) - f(x, y) : (x, y) \text{ is a forward-link in } \pi\}$ and $m^- = \min \{f(x, y) : (x, y) \text{ is a backward-link in } \pi\}$
 - (b) For each forward-link $(x, y) \in \pi$ increase $f(x, y)$ by m , and for each back-link $(x, y) \in \pi$ decrease $f(x, y)$ by m . Net-fbw increased = m .

(CONTD.)

A complex-augmentation path $\pi = \langle A, C, B, D \rangle$ with (B, C) a backward link; here, $m^+ = 1$, $m^- = 2$, and $m = 1$.



After flow-augmentation; the flowpath 1 along $\langle A, B, C, D \rangle$ is replaced by 1 along each of $\langle A, C, D \rangle$ and $\langle A, B, D \rangle$.

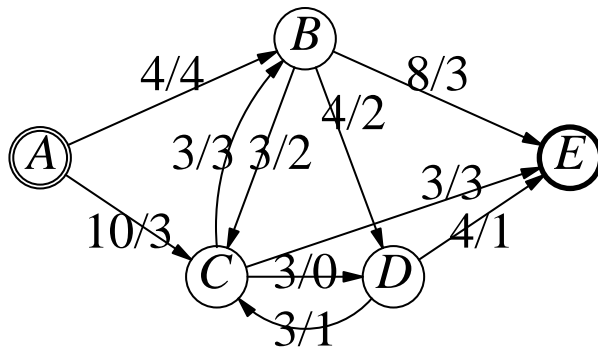
EXERCISE

1. Why is it true that we only need to consider complex flow-augmentation paths π that are acyclic? What advantage do we have by removing a cycle (if any) in π ?

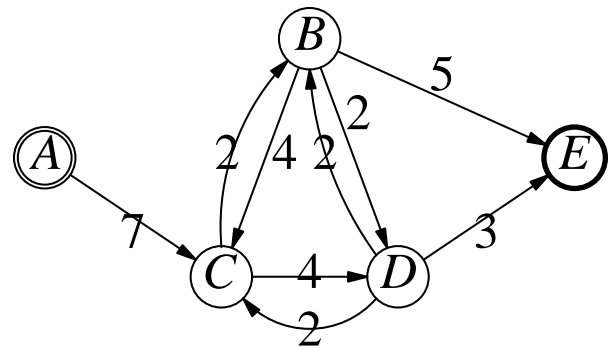
COMPUTATION OF OPTIMAL FLOW-AUGMENTATION PATH

Optimal FA-path $\langle s, x_1, \dots, x_n, d \rangle$:

- Gives the maximum increase in net-fbw.
- Assume that arcs to s and from d are eliminated, and the cyclic fbws involving those links have been eliminated.



(i) A network flow with net-fbw = 7.



(ii) The network G' for computing an optimal FA-path; shown are only $c'(x, y) > 0$.

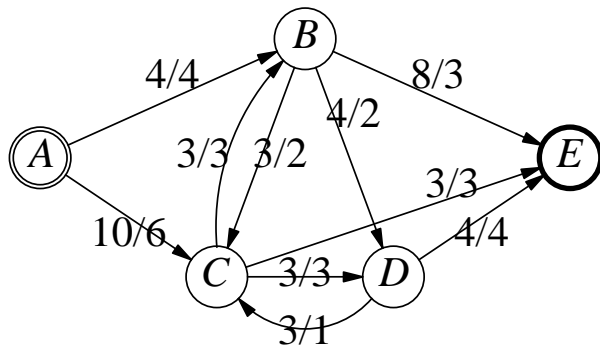
Link Capacities in G' :

- $c'(s, y) = c(s, y) - f(s, y) \geq 0$.
- $c'(x, d) = c(x, d) - f(x, d) \geq 0$.
- $c'(x, y) = [c(x, y) - f(x, y)] + f(y, x)$, for $x \neq s$ and $y \neq d$.

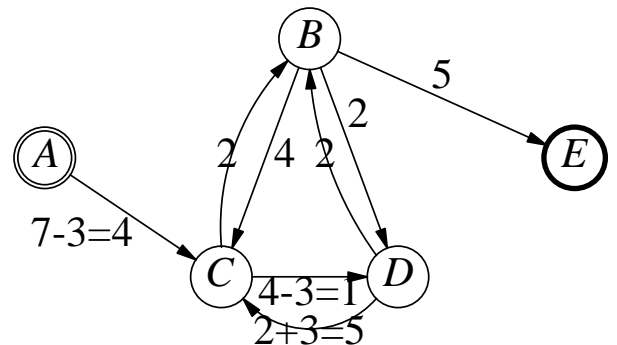
Optimal FA-path: $\pi = \langle A, C, D, E \rangle$; $C_{\min}(\pi) = 3$.

- The net-fbw is increased by $C_{\min}(\pi)$.
- It does not introduce any fbw to s or from d .

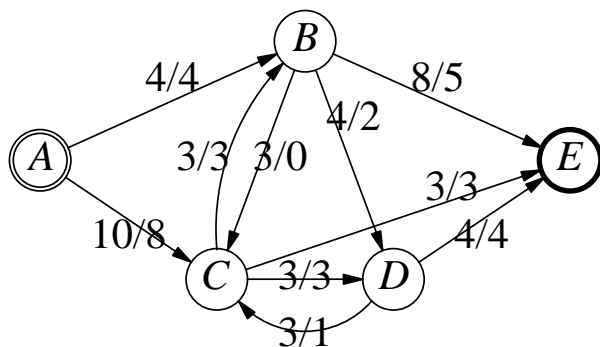
(CONTD.)



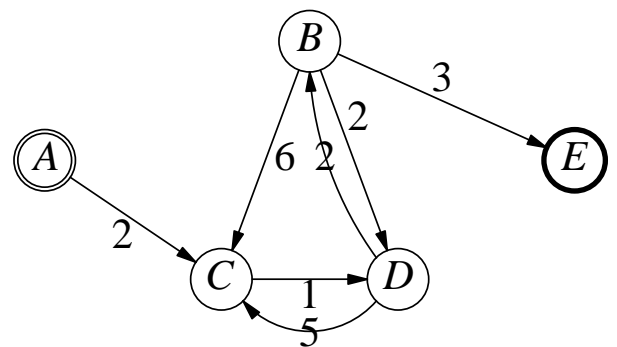
(iii) New net-fbw = $7+3 = 10$;
no need to eliminate cyclic
fbw 2 along $\langle C, D, C \rangle = 2$.



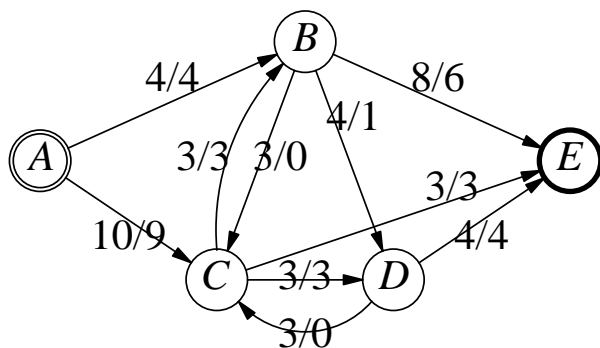
(iv) New G' and new optimal
FA-path $\pi = \langle A, C, B, E \rangle$
with $C_{\min}(\pi) = 2$.



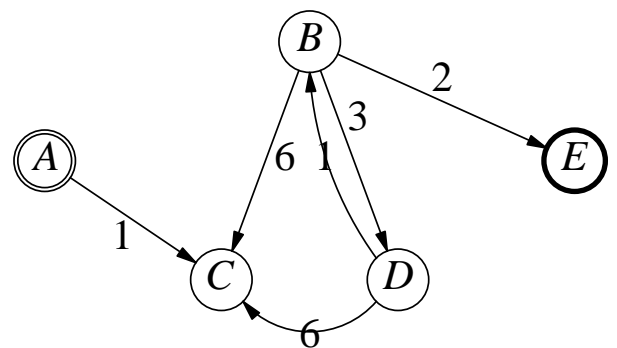
(v) New net-fbw = $10+2 = 12$.



(vi) New G' and new optimal
FA-path $\pi = \langle A, C, D, B, E \rangle$
with $C_{\min}(\pi) = 1$.



(vii) New net-fbw = $12+1 = 13$.



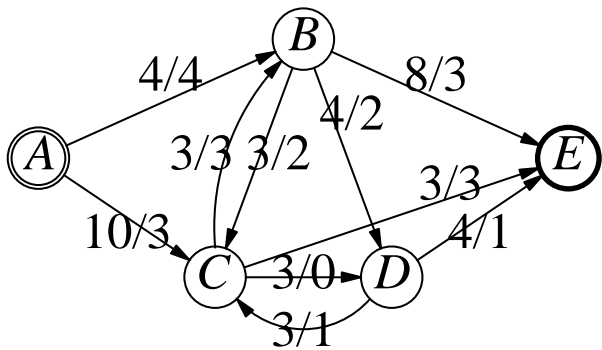
(viii) New G' ; there is no FA-path.
The current fbw 13 is maximum.

MAX-FLOW MIN-CUT THEOREM

Cut $(S, V - S)$: $S \subset V$ such that source $s \in S$ and sink $d \notin S$.

Value of cut $(S, V - S)$: $\text{val}(S, V - S) = \sum_{x \in S, y \in V - S} c(x, y)$

Example. Consider the network below, without the fbws for the moment. There are a total of $2^3 = 8$ cuts.



A network fbw;
net-fbw = 7.

- $S = \{A, C\}$: $v(S, V - s) = c(A, B) + c(C, B) + c(C, D) + c(C, E) = 13$.
- $S = \{A, B\}$: $v(S, V - s) = c(A, C) + c(B, C) + c(B, D) + c(B, E) = 25$.

Theorem. For any given fbw and any cut $(S, V - S)$, we have $\text{net-fbw} \leq \text{val}(S, V - S)$.

Proof. $\text{Net-fbw} = f^+(s) - f^-(s) = \sum_{x \in S} [f^+(x) - f^-(x)]$

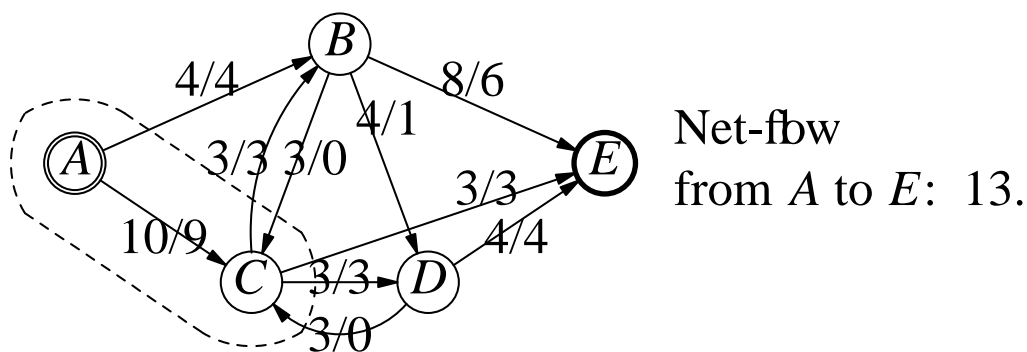
$$\begin{aligned}
 &= \sum_{x \in S, y \in V - S} f(x, y) - \sum_{y' \in V - S, x' \in S} f(y', x') \\
 &\leq \sum_{x \in S, y \in V - S} f(x, y), \text{ with equality if } \sum f(y', x') = 0 \\
 &\leq \sum_{x \in S, y \in V - S} c(x, y), \text{ with equality if } f(x, y) = c(x, y) \\
 &\hspace{15em} \text{for each } x \in S \text{ and } y \in V - S \\
 &= \text{val}(S, V - S)
 \end{aligned}$$

(CONTD.)

Coro. 1. $\text{Max-fbw} \leq \text{min-cut} = \min \{ \text{val}(S, V - S) : S \subset V \text{ and } S \neq \emptyset \}$. Moreover, if for some fbw f and some cut $(S, V - S)$, we have $\text{net-fbw} = \text{val}(S, V - S)$, then $\text{net-fbw} = \text{max-fbw}$ and $\text{val}(S, V - S) = \text{min-cut}$.

Coro. 2. If there is no fbw-augmentation path, then the fbw is the maximum.

Example. Consider the network fbw below, which is the same as (vii) on page 5.5 and for which there is no FA-path. Let $S = \{A, C\}$ = the nodes reachable from $s = A$ in G' .



Here, $f(x, y) = c(x, y)$ for each $x \in S$ and $y \in V - S$, and $f(y', x') = 0$ for each $x' \in S$ and $y' \in V - S$,

Hence, $\text{net-fbw} = \sum_{x \in S} [f^+(x) - f^-(x)] = \text{val}(S, V - S)$.

Thus, $\text{net-fbw} = \text{max-fbw} = \text{val}(S, V - S) = \text{min-cut}$.

EXERCISE

1. Give an example network to show that the following may not be true: if we always do the maximum fbw-augmentation

and f_1, f_2, \dots are the successive amount of fbw augmentations, then $f_1 \geq f_2 \geq \dots$.

2. Give an example network for which the maximum fbw augmentation may still create cycles of fbws.