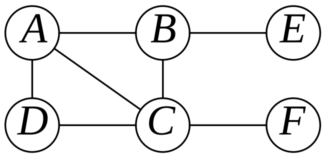
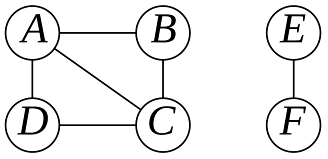
DEPTH-FIRST SEARCH OF A GRAPH

Some Applications:

- Finding a path between two nodes x and y (an xy-path).
- Finding if the graph is connected.
- Finding cut-vertices and bicomponents (maximal subgraph without a cut-vertex).



(i) A connected graph on nodes $\{A, B, \cdots, F\}$.



(ii) A disconnected graph on nodes $\{A, B, \cdots, F\}$.

Connected graph:

- There is a path between each pair of nodes x and y ($y \neq x$).

Question:

•? What are the simple (acyclic) DE-paths for the graph on left?

•? If there is path from some node z to every other node, then is it true that there is a path between every pair of nodes x and $y \neq x$?

•? Why is this result important?

Cut-vertex x**:**

- Removal of x and its adjacent edges destroys all paths (≥ 1) between some pair of nodes y and z; we say x separates y and z.

Question:

•? What are the cut-vertices in the above graphs.

•? What is the minimum number edges that need to be added to the first graph so that it has no cut-vertex.

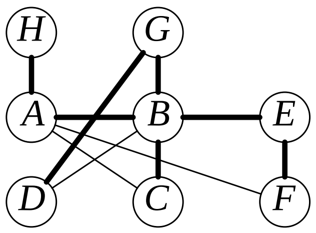
DEPTH-FIRST TREE

Spanning Tree (of a connected graph):

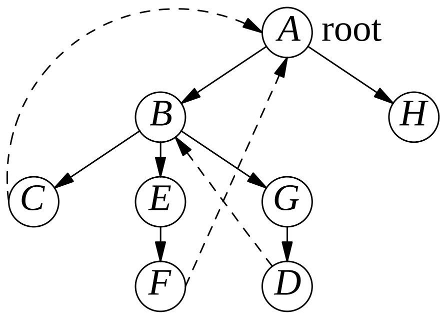
- Tree spanning all vertices (= n of them) of the graph.
- Each spanning tree has n nodes and $n - 1$ links.

Back-Edges and Cross-Edges (for a rooted spanning tree T):

- A non-tree edge is one of the following:
  - back-edge (x, y): joins x to ancestor $y \neq$ parent(x).
  - cross-edge (x, y): other non-tree edges.



(i) Solid lines show a spannig tree



(ii) Dashed lines show back-edges; no cross-edges. Tree edges directed from parent to child.

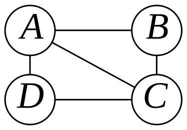
Depth-First Tree:

- A rooted spanning tree with no cross-edges (as in (ii) above).

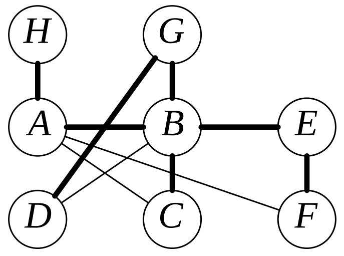
Question:

•? Show the back-edges and cross-edges if B is the root. Is this a depth-first tree (why)? Do the same for root = E and root = F.

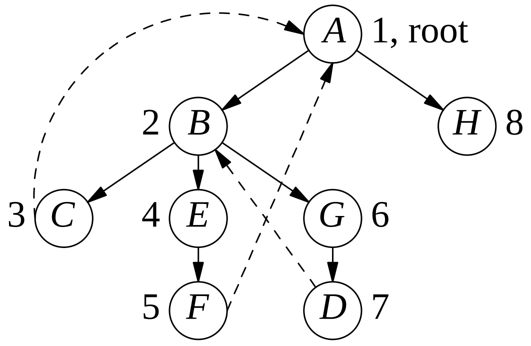
•? Show all spanning trees of the graph below (keep the nodes in the same relative positions).



VISITING EDGES IN A DEPTH-FIRST TRAVERSAL



adjList(A) = $\langle B, C, F, H\rangle$
adjList(B) = $\langle A, C, E, G, D\rangle$
adjList(C) = $\langle A, B\rangle$, adjList(D) = $\langle B, G\rangle$
adjList(E) = $\langle B, F\rangle$, adjList(F) = $\langle A, E\rangle$
adjList(G) = $\langle B, D\rangle$, adjList(H) = $\langle A\rangle$



(ii) The dfTree, dfLabels, and back-edges.

Order of Processing Links and Backtracking: StartNode = A.

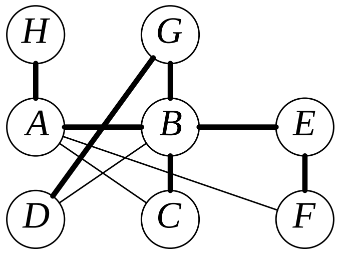
(A, B)	tree-edge	(G, B)	2nd visit
(B, A)	2nd visit	(G, D)	tree-edge
(B, C)	tree-edge	(D, B)	back-edge
(C, A)	back-edge	(D, G)	2nd visit
(C, B)	2nd visit	backtrack	$D \rightarrow G$
backtrack	$C \rightarrow B$	backtrack	$G \rightarrow B$
(B, E)	tree-edge	(B, D)	2nd visit
(E, B)	2nd visit	backtrack	$B \rightarrow A$
(E, F)	tree-edge	(A, C)	2nd visit
(F, A)	back-edge	(A, F)	2nd visit
(F, E)	2nd visit	(A, H)	tree-edge
backtrack	$F \rightarrow E$	(H, A)	2nd visit
bacltrack	$E \rightarrow B$	backtrack	$H \rightarrow A$
(B, G)	tree-edge	backtrack	$A \rightarrow$

Two Basic Operations: Move forward and backtrack.

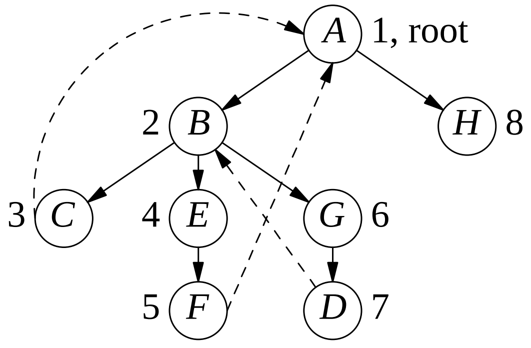
Question:

•? Show the order of processing the links and backtracking if we reorder adjList(A) = $\langle F, B, C, H\rangle$ for the above graph.

USE OF STACK AND DEPTH-FIRST LABEL FOR DEPTH-FIRST TRAVERSAL



adjList(A) = ⟨B, C, F, H⟩
adjList(B) = ⟨A, C, E, G, D⟩
adjList(C) = ⟨A, B⟩, adjList(D) = ⟨B, G⟩
adjList(E) = ⟨B, F⟩, adjList(F) = ⟨A, E⟩
adjList(G) = ⟨B, D⟩, adjList(H) = ⟨A⟩



(ii) The dfTree, dfLabels, and back-edges.

Stack	Node or Link	LastDf-Label	Processing
⟨A⟩	A	0 → 1	dfLabel(A) = 1, nextNode = B = adjList(A, 0), nextToVisit(A) = 1 ← 0
	(A, B)		tree-edge, parent(B) = A, add B to stack
⟨A, B⟩	B	1 → 2	dfLabel(B) = 2, nextNode = A = adjList(B, 0), nextToVisit(B) = 1 ← 0
	(B, A)		second-visit, nextNode = C = adjList(B, 1), nextToVisit(B) = 2 ← 1
	(B, C)		tree-edge, parent(C) = B, add C to stack
⟨A, B, C⟩	C	2 → 3	dfLabel(C) = 3, nextNode = A = adjList(C, 0), nextToVisit(C) = 1 ← 0
	(C, A)		back-edge, nextNode = B = adjList(C, 1), nextToVisit(C) = 2 ← 1,
	(C, B)		second-visit, backtrack
⟨A, B⟩	B		nextNode = E = adjList(B, 2), nextToVisit(B) = 3 ← 2
	(B, E)		tree-edge, parent(E) = B, add E to stack
⟨A, B, E⟩	E	3 → 4	dfLabel(E) = 4, nextNode = B = adjList(E, 0), nextToVisit(E) = 1 ← 0
	(E, B)		second-visit, nextNnode = F = adjList(E, 1), nextToVisit(E) = 2 ← 1
	(E, F)		tree-edge, parent(F) = E = adjList(E, 1), nextToVisit(E) = 3 ← 2
...	...	...	...

DATA-STRUCTURES IN DEPTH-FIRST TRAVERSAL ALGORITHM

Array Data-Structures (each array-size = numNodes):

dfLabels[*i*]:	the depth-first label (≥ 1) assigned to node *i*; each dfLabels[*i*] = 0, initially.
nextToVisits[*i*]:	nextToVisits[*i*] gives the position of the item in adjList of node *i* that is to be visited next from node *i*; each nextToVisit[*i*] = 0, initially;
parents[*i*]:	the parent of node *i* in depth-first tree; parents[*startNode*] = *startNode*, initially.

Additional Data-Structures:

- lastDfLabel = 0, initially;
  it is incremented by one *before* assigning it to a node.
- stack[0..numNodes-1];
  it is initialized with the startNode and it contains the path in the depth-first tree from the root to the current node = top(stack).

Notes:

- You can add parent, dfLabel, and nextToVisit information in the structure GraphNode as shown below.

```
typedef struct {
  int degree, *adjList, //arraySize = degree
      nextToVisit, //0<=nextToVisit<degree
      parent, dfLabel;
} GraphNode;
```

In that case, the next link to visit from node *i* is link (*i*, *j*), where *j* = nodes[*i*].adjList[nodes[i].nextToVisit]; otherwise, *j* = nodes[*i*].adjList[nextToVisit[i]].

PSEUDOCODE FOR DEPTH-FIRST TRAVERSAL ALGORITHM

First Version:

1. Try to go down the tree (which is being created) from the current node *x* by choosing a link (*x*, *y*) in the graph from *x* to a node *y* not yet visited and adding the link (*x*, *y*) to the tree.
2. If there is no such node *y*, then backtrack to the parent of the current node, and stop when you backtrack from the startNode (= root of the tree).

A More Detailed Version (with parents[] and nextToVisit[]):

1. Initialize the arrays parents[] and nextToVisit[] as indicated in the previous page and let currentNode = startNode.
2. Search adjList[currentNode] from the position nextToVisit[currentNode] onwards for a nextNode which is not yet visited,
3. [Move forward.] If (nextNode is found), then let parents[nextNode] = currentNode, update nextToVisit[currentNode], and currentNode = nextNode.
4. [Backtrack.] If (no nextNode is found) then do the following:
   - (i) If (currentNode = startNode) then stop.
   - (ii) Otherwise backtrack, i.e., let currentNode = parents[currentNode] and goto step (2).

DEPTH-FIRST TRAVERSAL ALGORITHM

Algorithm DepthFirstTraverse:

Input: A graph in adj-list form and a start-node.
Output: A depth-first spanning tree for the connected component containing start-node.

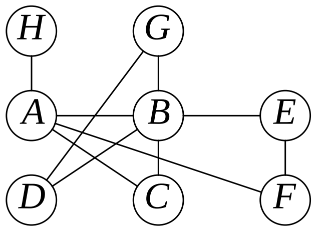
1. Initialize lastDfLabel, the stack, and also the arrays dfLabels, parents, and nextToVisits as indicated in the previous page.
2. While (stack $\neq$ empty) do the following:
   - (a) Let currentNode = top(stack) and if (dfLabels[currentNode] = 0) then update lastDfLabel and let dfLabels[currentNode] = lastDfLabel.
   - (b) If (nextToVisit[currentNode] = degree[currentNode]) then backtrack by removing top of stack.
   - (c) Otherwise, let nextNode = the node in position nextToVisit[currentNode] in adjList(currentNode), and update nextToVisit[currentNode].

     Classify the link (currentNode, nextNode) as follows searching for a tree-edge (if any) from currentNode:
     - (i) if (dfLabels[nextNode] = 0) then it is a tree-edge; let parent[nextNode] = currentNode, add nextNode to stack.
     - (ii) if (dfLabels[nextNode] < dfLabels[currentNode]) then if (nextNode $\neq$ parents[currentNode]) then it is a back-edge and otherwise it is a second visit of tree-edge.
     - (iii) Otherwise, it is a second visit to a back-edge.

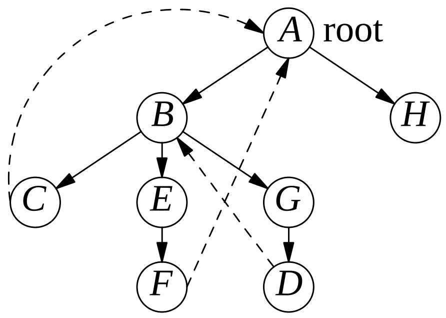
Note: We merged old (c)-(d), eliminated all goto's and putting (c)-(d) in a loop, and split d(ii) into d(ii)-(iii). Also, added an if-part in (a).

EXERCISE

1. How many times a link (x, y) is visited and when?
2. What is the total number of back-edges?
3. How many tree-edges and back-edges are there at a node x?
4. How many times does a node x enter the stack and when?
5. How many times do we backtrack to a node x? How many times do we backtrack from a node x and when?
6. Shown below is a graph and a depth-first tree for startNode = A.



(i) A graph.



(ii) A depth-first tree.

   (a) Show all possible orderings of the adjacency list of A which will give this tree when we apply the depth-first traversal algorithm. Do the same for node B, keeping startNode = A.

   (b) Give the total number of ways of ordering the various adjacency lists which will give the above depth-first tree.

   (c) Arrange all adjacency-lists in such a way that at each node x, the tree-edges from x to its children are visited first, then the back-edges from x (if any, to its ancestors), next the second-visit of tree-edge connecting x to its parent (if any), and finally the second-visit of back-edges to x (if any, from its descendants).

7. How do you distinguish a second-visit to a tree edge vs. a second-visit to a backedge?
8. For each edge processed in the table on page 3, show the associated step of the algorithm on page 7 that is used for it.
9. Show the modifications to the delth-first traversal algorithm to compute maxDfLabel(x) = max {dfLabel(y): y = x or a descendant of x} for each node x. Use the depth-first tree above to explain how this works. How can you compute the size of the subtree at x using maxDfLabel(x)?
10. Is it true that the nodes in the subtree at x of a depth-first tree consists of all nodes that can be reached from x without using any of the other nodes on the path from root to x?

A RECURSIVE VERSION OF DF-TRAVERSAL

Notes:

- Backtracking corresponds to return from the recursion.
- The lastDfLabel and the dfLabels-array can be *static* in DF-TRAVERSAL; the nextToVisit-array is not used.
- The recursive-call at x creates the subtree of df-tree at x as we build the df-tree via child-pointers, etc.

Algorithm DF-TRAVERSAL(currNode): //first call: currNode = s

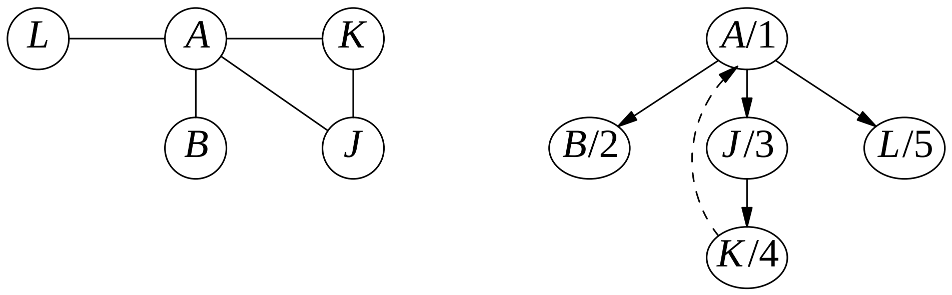
Input: A graph in adj-List form and a start-node s.
Output: The df-tree, with root = s.

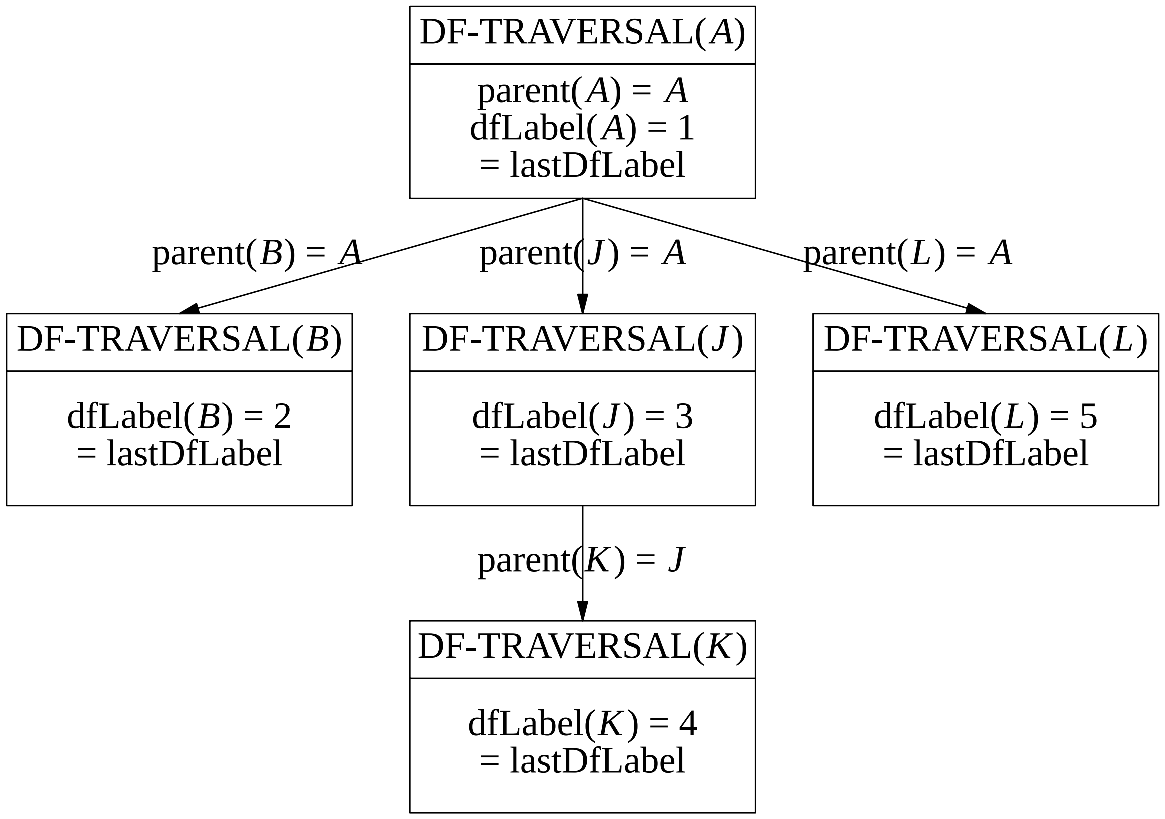
```
1. [Create df-tree node.] Create currTreeNode in df-tree for currNode. If (lastDfLabel = 0), then initialize dfLabel($x$) = 0 for each $x$ and let parent(currTreeNode) = currTreeNode. Now, let dfLabel(currNode) = lastDfLabel = lastDfLabel + 1.
2. For (each node $x$ in adjList(currNode)) do the following:
       If (dfLabel($x$) = 0) then add the root-node (which corresponds to $x$) of the subtree returned by DF-SEARCH($x$) as the next child of currTreeNode.
3. Return(currTreeNode).
```

Complexity: $O(|V| + |E|) = O(|E|)$ for a *connected* G.

- An edge (x, y) is examined at most once in each direction.
- A constant amount of work per node x (assigning dfLabel(x), adding x to the df-tree, and backtracking from x).

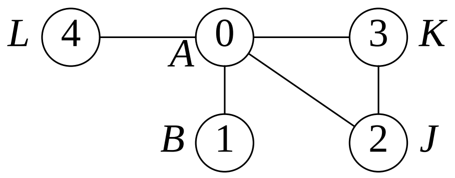
ILLUSTRATION OF THE RECURSIVE DF-TRAVERSAL





- #(calls to DF-TRAVERSAL) = #(nodes in df-tree) = #(nodes in graph), if G is connected.
- #(direct calls from DF-TRAVERSAL(x)) = #(children x).

A SAMPLE OUTPUT OF DF-TRAVERSAL PROGRAM



```
Input Graph:
numNodes = 5
0 (4): 1 2 3 4
1 (1): 0
2 (2): 0 3
3 (2): 0 2
4 (1): 0
------------------------------
startNode = 0, dflabel(0) = 1
processing (0,1): tree-edge, dfLabel(1) = 2
processing (1,0): tree-edge 2nd-visit
backtracking 1 -> 0
processing (0,2): tree-edge, dfLabel(2) = 3
processing (2,0): tree-edge 2nd-visit
processing (2,3): tree-edge, dfLabel(3) = 4
processing (3,0): back-edge
processing (3,2): tree-edge 2nd-visit
backtracking 3 -> 2
backtracking 2 -> 0
processing (0,3): back-edge 2nd-visit
processing (0,4): tree-edge, dfLabel(4) = 5
processing (4,0): tree-edge 2nd-visit
backtracking 4 -> 0
------------------------------
```

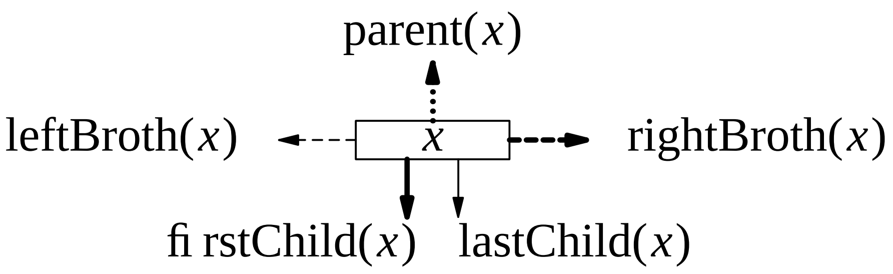
PROGRAMMING EXERCISE

1. Create a function depthFirstTraverse(int startNode). It should produce output as indicated above. Use an auxiliary function readGraph (input fi le should be in the same form as for a digraph using adjacency-lists), which should output the graph read. keep the adjacency lists sorted in increasing order. Show the output for the 8 node graph on page 2.2 with startNode *B* (= 1).

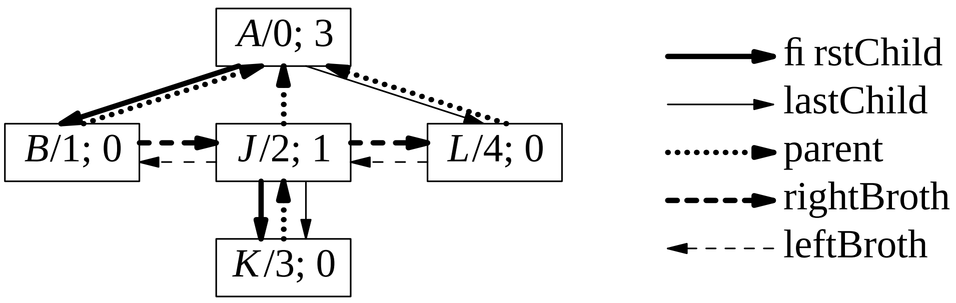
NODE-STRUCTURE FOR AN ORDERED-TREE

- The parent-links alone do not allow us to traverse the df-tree from the root (= startNode of depth-first traversal).
- Since the number of children of a node is known ahead of time, the children are kept in a linked-list.
- The lastChild pointer helps addition of a child and firstChild pointer accessing the children.

```
typedef struct TreeNode {
   int node, numChildren;
   struct TreeNode *parent,
                   *leftBroth, *rightBroth,
                   *firstChild, *lastChild;
} TreeNode;
```

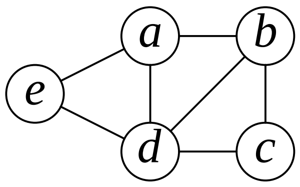


Example. Null-pointers not shown; firstChild(J) = lastChild(J). The nodes are numbered 0, 1, ⋯ in alpabetical order.

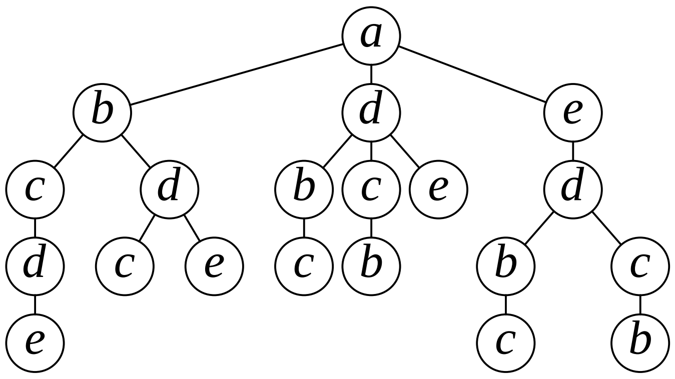


ALL ACYCLIC PATHS IN A GRAPH FROM A START-NODE

Example. Shown below is a graph and the tree of all acyclic paths from a.



(i) A graph G.



(ii) The tree of acyclic paths from a in G.

Some Properties of this Tree (startNode = s):

- A node x appears in the tree as many times as the number of acyclic paths to x from s.
- We can find all cycles (without repetition of nodes) at a node s from these paths:
  - Combine an sx-path with the edge (s, x), if exists.
  - Each cycle is obtained twice in this process, once in each direction.
- We can find if there is a Hamiltonian cycle (cycle containing all nodes exactly once).

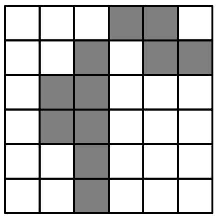
Note:

- To deremine all acyclic paths from startNode, use the depth-first traversal with the change: reset nextToVisit[i] = 0 = dfLabel[i] in backtrack from node i.

OBJECT-REGION IN A BINARY-IMAGE

Binary-image:

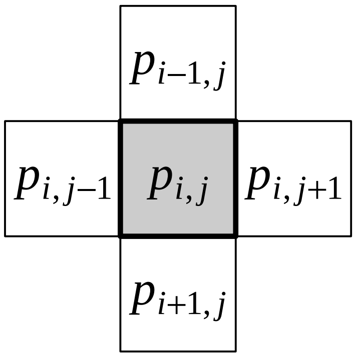
- An $m \times n$ array, where a pixel $p_{ij} = 1$ (shown below as a shaded square) represents a part of an *object* in the image; a pixel $p_{ij} = 0$ represents a part of the *background*.



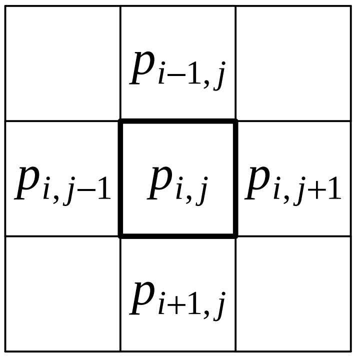
A 6×6 binary image.

Adjacent Pixels:

- Two 1-pixels are adjacent if they share an edge.
- Two 0-pixels are adjacent if they share a corner or an edge.



At most 4 neighbors of an 1-pixel p_{ij}.



At most 8 neighbors of a 0-pixel p_{ij}.

Image-object: A *maximal* set of connected 1-pixels p_{ij}.

Example. The above image has 2 objects of sizes 7 and 4 and two background regions of sizes 1 and 24.

Question: Give a 5×5 image with maximum number of objects.

OBJECT DETERMINATION BY DF-TRAVERSAL

Order of Visit of Neighbors of 1-pixel p_{ij} (if present):

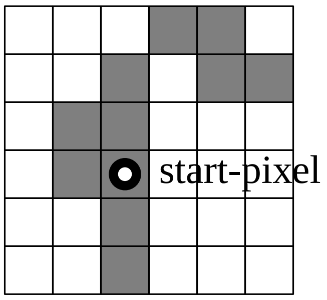
(1) $p_{i,j+1}$ (E-neighbor) (3) $p_{i,j-1}$ (W-neighbor)
(2) $p_{i-1,j}$ (N-neighbor) (4) $p_{i+1,j}$ (S-neighbor)

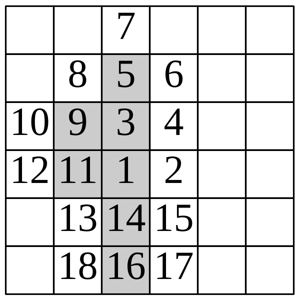
Additional Backtracking Rule:

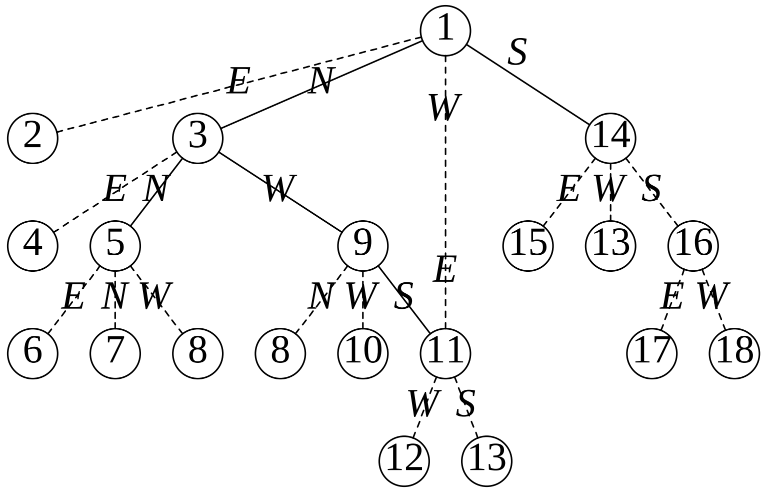
- If we reach a 0-pixel, we immediately backtrack.

Example. Shown below is the depth-first labeling of the pixels visited starting at the position labeled 1.

All neighboring 0-pixels of the region R containing the start-pixel are also visited. The total number of visits to those 0-pixels is at most $2(|R| + 1)$ - why?





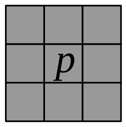


The dashed lines indicate neighbor that are visited but are not in the current region. These nodes appear as many times as they are visited, each time causing an immediate backtracking; both nodes 8 and 13 appear here twice.

INTERNAL AND EXTERNAL BOUNDARIES OF AN OBJECT

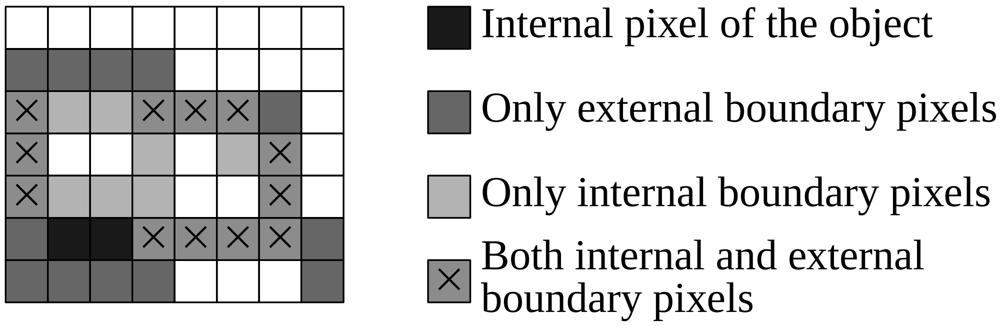
Non-Boundary (internal) Pixel *p* of An Object:

- Completely surrounded by *8-neighbor* object pixels.



Two Types of Background Pixels:

- 0-regions are based on 8-neighbors.
- *External regions:* 0-regions which contain at least one pixel in the first or last row or first or last column.
- *Holes:* other 0-regions completely surrround by object pixels; does not contain pixels from first (last) row or column.


Two holes of sizes 2 and 3.
Two external regions of sizes 3 and 15.
Internal pixel of the object
Only external boundary pixels
Only internal boundary pixels
Both internal and external boundary pixels

Two Types of Boundary (non-internal) Pixels:

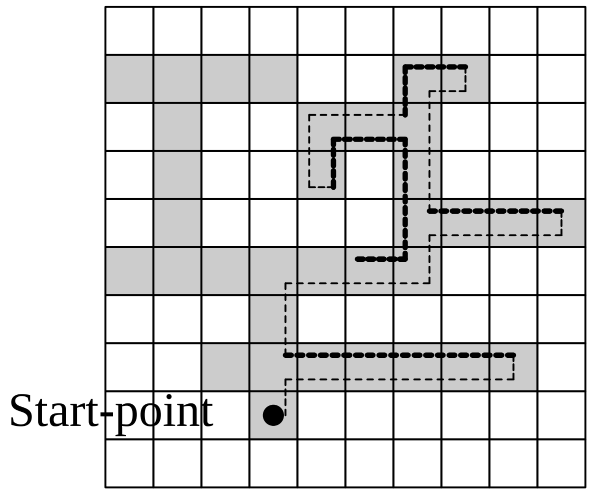
- *Internal boundary:* Objects pixels adjacent to holes (based on 8-neighbors).
- *External boundary:* Object pixels adjacent (based on 8-neighbors) to external regions or in the first (last) row or column.

Question: Show the boundary and internal pixels if the pixel $p_{5,4}$ above were a 0-pixel (merging two holes into one),

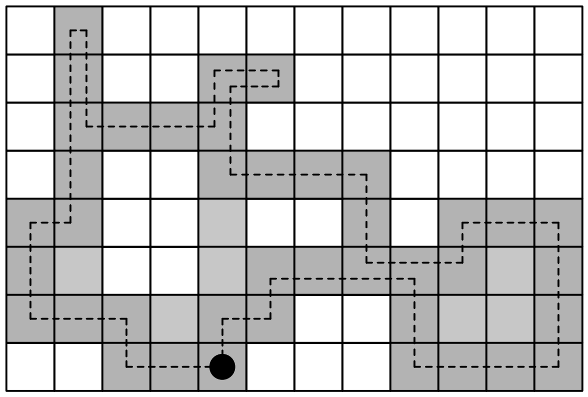
VISITING A MAZE OR EXTERNAL BOUNDARY OF AN OBJECT

Strategy:

- Keep to the right, starting at a southmost pixel of maze or object.
- Assume that we arrived at the start-position with a north-move (thus, attempt to go east first at start-position).



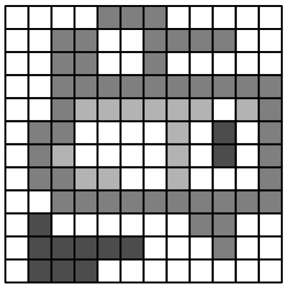
Part of the move sequence visiting a maze.
The trail of "return visits" parts are shown in bold.



The move sequence visiting the external boundary of an object.

Question: Show the move sequence if we start at pixel $p_{3,3}$.

OBJECT SIMPLIFICATION BY FILLING UP HOLES IN THE OBJECT

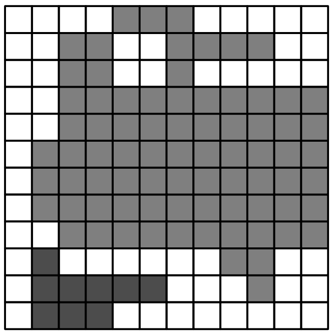


- An image with three objects.
- The largest object contains two holes and one of the objects is placed in one of those holes.
- The non-boundary pixels of this object are shown in light shade.

External Regions:

- The 0-regions which contain at least one 0-pixel from any of top/bottom rows and leftmost/rightmost columns.

Holes: These are 0-regions other than the external regions.



- After filling-in the two holes in the largest object.
- This swallows the third object in one of those holes.

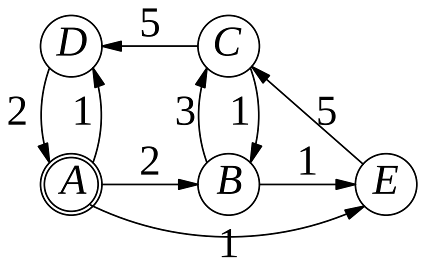
Questions:

1? Show df-labels of pixels in the 0-region in the top image starting at $p_{1,1}$ (top-left pixel); do not label 1-pixels that may be visited.

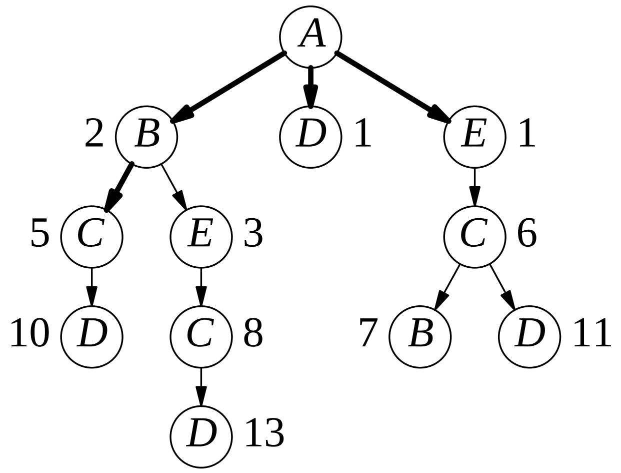
2? Give a psuedocode to detect the pixels in the holes of an object. What is the computational complexity?

3? Mark successively as 1, 2, $\cdots$ for the first visit to the 1-pixels starting at the southmost pixel $p_{11,10}$ ($p_{1,1}$ = the top-left pixel) of the largest object in the top figure and using the rule "keep to the right". The next pixel visited would be $p_{10,10}$. (Note: there is no backtracking here, and some pixel may be visited more than once. As usual, assume we arrived at $p_{11,10}$ by going north.)

TREE OF SHORTEST-PATHS IN DIGRAPHS



- A digraph $\vec{G}$ with each $c(x, y) \geq 0$.
- Start-node $s = A$.



- Bold links show the tree of shortest-paths to various nodes.

The tree of acyclic paths from A; shown next to each node is the length of the path from root = A.

Some Important Observations:

- Any subpath of a shortest path is a shortest path.
- The shortest paths from a startNode to other nodes can be chosen so that they form a tree.

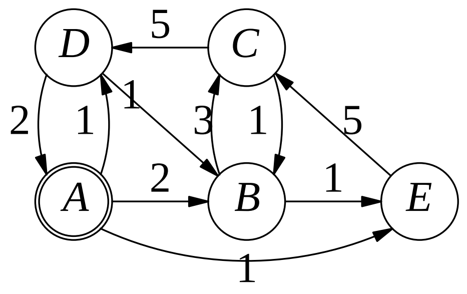
Question:

•? What are some minimum changes to the link-costs that will make $\langle A, B, E, C\rangle$ the shortest AC-path?

•? Show the new tree of acyclic paths ad the shortest paths from startNode = A after adding the link (D, B) with cost 1.

•? Also show the tree of acyclic paths and the shortest paths from the startNode = D.

ILLUSTRATION OF DIJKSTRA’S SHORTEST-PATH ALGORITHM



- $d(x)$ = length of best path known to x from the startNode = A.
- A node is closed if shortest path to it known.
- A node is OPEN if a path to it known, but no shortest path is known.

"?,?" indicates unknown values, "···" indicates no changes, and "–" indicates path-length not computed (would not have changed any way).

Open	Node	Links	$d(x)$ and parent(x)				
Nodes	Closed	processed	A	B	C	D	E
$\{A\}$	$\varnothing$		0, A	?, ?	?, ?	?, ?	?, ?
$\{B\}$	A	(A, B)		2, A			
$\{B, D\}$		(A, D)				1, A	
$\{B, D, E\}$	(A, E)					1, A	
$\{B, E\}$	D	(D, A)	–				
		(D, B)		···			
$\{B, C\}$	E	(E, C)			6, E		
$\{C\}$	B	(B, C)			5, B		
		(B, E)					–
$\varnothing$	C	(C, B)		–			
		(C, D)				–	

Question:

•? List all paths from A that are looked at (length computed) above.

•? When do we look at a link (x, y)? How many times do we look at a link (x, y)?

•? What might happen if some $c(x, y) < 0$?

DIJKSTRA'S ALGORITHM

Terminology (OPEN$\cap$CLOSED = $\emptyset$):

CLOSED = $\{x$: $\pi_m(s, x)$ is known$\}$.
OPEN = $\{x$: some $\pi(s, x)$ is known but $x \notin$ CLOSED$\}$.

Algorithm DIJKSTRA (shortest paths from s):

Input:	AdjList(x) for each node x in $\vec{G}$, each $c(x, y) \geq 0$, a start-node s, and possibly a goal-node g.
Output:	A shortest sg-path (or sx-path for each x reachable from s).

1. [Initialize.] $d(s) = 0$, mark s OPEN, parent(s) = s (or NULL), and all other nodes are unmarked.
2. [Choose a new closing-node.] If (no OPEN nodes), then there is no sg-path and stop. Otherwise, choose an OPEN node x with the smallest $d(\cdot)$, with preference for $x = g$. If $x = g$ or all but one node are closed, then stop.
3. [Close x and Expand $\pi_m(s, x)$.] Mark x CLOSED and for (each $y \in$ adjList(x) and y not marked CLOSED) do:
   if (y not marked OPEN or $d(x)+c(x, y) < d(y)$) then let parent(y) = x, $d(y) = d(x) + c(x, y)$, and mark y OPEN.
4. Go to step (2).

Complexity: $O(N^2)$.

- A node x is marked CLOSED at most once and hence a link (x, y) is processed at most once.
- Each iteration of steps (2) and (3) takes $O(N)$ time.