TEMPORAL LOGIC

Boolean proposition: A variable x with values T/F.

Temporal proposition:
A variable whose value is a sequence of T/F corresponding to time $t = 1, 2, 3, \cdots$: $x = (\mathrm{F, F, T, F}, \cdots)$.

Temporal operators:
Each operator combines one or more temporal propositions to give another temporal proposition.

(1) Extension of boolean operations

(a) $A \vee B$, $A \wedge B$, $\neg A$: each defined coordinate-wise; e.g., $[A \wedge B](t) = A(t) \wedge B(t)$ for all $t \geq 1$.

$$\begin{array}{rl} A & = (\mathrm{T, F, F, T, F, T}, \cdots) \\ B & = (\mathrm{T, T, F, T, F, T}, \cdots) \\ \hline A \vee B & = (\mathrm{T, T, F, T, F, T}, \cdots) \\ A \wedge B & = (\mathrm{T, F, F, T, F, T}, \cdots) \\ \neg A & = (\mathrm{F, T, T, F, T, F}, \cdots) \end{array}$$

(b) $A \rightarrow B = \neg A \vee B$, $A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A)$

(2) New operations (U has higher priority than $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$; unary operators have higher priority than others)

(c) $\bullet$ (next-time operator): $[\bullet A](t) = A(t+1)$ for all $t \geq 1$.

$$\begin{array}{rl} A & = (\mathrm{T, F, F, T, F, T}, \cdots) \\ \bullet A & = (\mathrm{F, F, T, F, T}, \cdots) \end{array}$$

$$\begin{array}{rl} \bullet(A \wedge B) & = \bullet A \wedge \bullet B \\ \bullet(A \vee B) & = \bullet A \vee \bullet B \\ \bullet(\neg A) & = \neg \bullet A \end{array}$$

(d) $\blacklozenge$ (sometimes operator): $[\blacklozenge A](t) = \exists t'[(t' \geq t) \wedge A(t')]$.

$$\begin{array}{rl} A & = (\mathrm{T, F, F, T, F, T, F, F}, \cdots), \text{ all F in the part "}\cdots\text{"} \\ \blacklozenge A & = (\mathrm{T, T, T, T, T, T, F, F}, \cdots) \end{array}$$

$$\begin{array}{rl} \blacklozenge(A \wedge B) & \neq \blacklozenge A \wedge \blacklozenge B \text{ (but } \blacklozenge(A \wedge B) \rightarrow \blacklozenge A \wedge \blacklozenge B \text{ is a tautology)} \\ \blacklozenge(A \vee B) & = \blacklozenge A \vee \blacklozenge B \\ \bullet\blacklozenge A & = \blacklozenge\bullet A \end{array}$$

(e) $\Box$ (always operator): $[\Box A](t) = \forall t'[(t' \geq t) \rightarrow A(t')]$.

$$\begin{array}{rl} A & = (\mathrm{T, F, F, T, F, T, T, T}, \cdots), \text{ all T in the part "}\cdots\text{"} \\ \Box A & = (\mathrm{F, F, F, F, F, T, T, T}, \cdots) \end{array}$$

$$\begin{array}{rl} \Box(A \wedge B) & = \Box A \wedge \Box B \\ \Box(A \vee B) & \neq \Box A \vee \Box B \text{ (but } \Box A \vee \Box B) \rightarrow \Box(A \vee B) \text{ is a tautology)} \\ \bullet\Box A & = \Box\bullet A \\ \neg\Box A & = \blacklozenge\neg A \end{array}$$

(e) U (until operator): $[AUB](t) = [\Box A](t) \vee \exists t'[(t' \geq t) \wedge B(t') \wedge \forall t''((t'' < t') \rightarrow A(t''))]$.

$$\begin{array}{rl} A & = (\mathrm{T, F, F, T, F, T, T, T, F, T}, \cdots), \text{ all T in the part "}\cdots\text{"} \\ B & = (\mathrm{F, F, T, T, F, F, F, T, F, F}, \cdots), \text{ all F in the part "}\cdots\text{"} \\ \hline AUB & = (\mathrm{F, F, T, T, F, T, T, T, F, T}, \cdots), \text{ all T in the part "}\cdots\text{"} \end{array}$$

EXERCISE

1. Which of the following is true? For each case which is false, give an example to illustrate.

(i)	$(A{\wedge}B)UC = (AUC) \wedge (BUC)$	(i′)	$(A{\vee}B)UC = (AUC) \vee (BUC)$
(ii)	$AU(B{\wedge}C) = (AUB) \wedge (AUC)$	(ii′)	$AU(B{\vee}C) = (AUB) \vee (AUC)$
(iii)	$AU(\blacklozenge B) = \blacklozenge(AUB)$	(iii′)	$(\blacklozenge A)UB = \blacklozenge(AUB)$
(iv)	$AU(\Box B) = \Box(AUB)$	(iv′)	$(\Box A)UB = \Box(AUB)$
(v)	$(\blacklozenge A)U(\blacklozenge B) = \blacklozenge(AUB)$		
(vi)	$(\Box A)U(\Box B) = \Box(AUB)$		
(vii)	$(\bullet A)U(\bullet B) = \bullet(AUB)$		
(vii)	$(\neg A)U(\neg B) = \neg(AUB)$		

2. Simplify the following temporal formulas as much as possible.

(i)	$\blacklozenge\blacklozenge A$	(ii)	$\Box\Box A$	(iii)	
(v)	$\blacklozenge(A \vee \bullet\blacklozenge A)$	(vi)	$\blacklozenge(A \wedge \bullet\blacklozenge A)$	(vii)	

3. For each of the following temporal formulas, state in ordinary English the property described by the formula (i.e., the property should characterise all situations that satisfy the formula). For eample, (i) describes the property "A occurs infinitely often".

(i) $\Box\blacklozenge A$ (ii) $\blacklozenge\Box A$

(iii) $\Box(A \vee \bullet\Box A)$

(iv) $\Box(A \rightarrow \bullet\Box\neg A)$ (v) $\blacklozenge(A{\wedge}\bullet\blacklozenge A)$

(vi) $(\neg BUA) \wedge (\neg AUB) \wedge \Box(B \rightarrow \bullet\Box\neg B)$

(vi) $(A{\wedge}\bullet\neg A{\wedge}\bullet\bullet\neg A) \wedge \Box((\neg A{\wedge}\bullet\neg A) \rightarrow \bullet\bullet(A{\wedge}\bullet\neg A{\wedge}\bullet\bullet\neg A))$

4. It is easy to see that the following are true.

$\phi_1(A) = \blacklozenge(A \wedge \bullet\Box\neg A)$ expresses the fact "A occurs al least once" (see (v) above)

$\phi_2(A) = \blacklozenge(A \wedge \bullet\phi_1(A))$ expresses the fact "A occurs at least twice"

What does the formula $\psi_2(A) = \phi_1(A){\wedge}\neg\phi_2(A)$ express? How about the formula $\blacklozenge A{\wedge}\Box(A \rightarrow \bullet\Box\neg A)$? What is the formula $\psi_2(A)$ for "A occurs exactly twice"? Give an example of A to show that $\blacklozenge(A{\wedge}\psi_2(A))$ does not capture the fact "A occurs at three times".

5. Convert each of the following English statements into an equivalent temporal formula.

(i) There are no consecutive occurrences of A.

(ii) A occurs at least once in every weak (the first weak begins at $t = 1$).

(iii) Every run of A is of length 3 (a run consists of a maximum contiguous period during which A occurs).

(iv) Every new occurrence of A is immediately preceded by at least two new occurrences of B.

(v) A and B occurs the first time simultaneously (in particular, they may never occur at all).

(vi) A is true until the second occurrence of B (in particular, if B occur less than twice, then A has to be true all the time).

(vii) B occurs exactly at the first occurence of A, if any (in other words, if A never occurs, then neither does B).

(viii) B occurs exactly at the every second occurrence of A.

$\phi_1(A, B) =$ A formula for "A and B happens the first time at the same instance".
$\phi_2(B) =$ A formula for "B happens at most once".
$\phi_3(A, B) =$ A formula for "B happens just at the first occurence of A".
$= \phi_1(A, B){\wedge}\phi_2(B)$
$\phi_4(A, B, C) =$ A formula for "B happens exactly at the second occurrence of A".
$= \phi_3(A, C) \wedge \phi_3(A{\wedge}\neg C, B)$

A TEMPORAL FORMULA FOR "IT RAINS SOMETIMES DURING EACH HURRICANE"

1. We have a hurricane and it will rain all the time during hurricane: $H \wedge [RU\neg H]$
2. We have a hurricane and it will remain dry all the time during hurricane: $H \wedge [\neg RU\neg H]$
3. We have a hurricane and it will not remain dry all the time during hurricane, i.e., it will rain sometime during the hurricane.

$$H \wedge \neg[\neg RU\neg H] \quad \text{or simply} \quad \neg[\neg RU\neg H]$$

because any situation that makes the second formula true will have $H = T$ at time 1.

4. What does the following formula translate to?

$$\Box(\neg[\neg RU\neg H]) = \neg\blacklozenge[\neg RU\neg H]$$

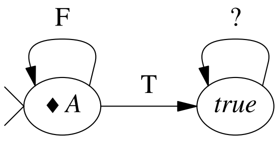
5. What does the following formula translate to (note that the first H is not needed because to make the formula true one has to have $H = T$ at time 1)?

$$H \wedge \neg[\neg RU\neg H] \wedge \Box[\neg H \wedge \bullet H \rightarrow \bullet\neg(\neg RU\neg H)]$$

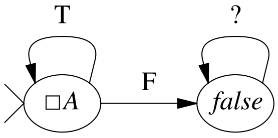
6. The following formula describes that there is a hurricane (now, in day 1) and there are at least two rainy days in this hurricane duration. It uses the auxiliary temporal proposition S to capture the starting day of the rain. Since S occurs exactly once, namely, at the first occurrence of R, $R\wedge\neg S$ kills only the first occurrence of R.

$$\neg[\neg RU\neg H] \wedge [(\neg SUR)\wedge(\neg RUS)\wedge\Box(S\rightarrow\bullet\Box\neg S)] \wedge \neg[\neg(R\wedge\neg S)U\neg H)]$$

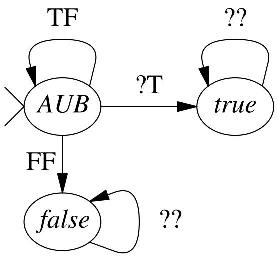
EXAMPLES OF STATE-GRAPH FOR TEMPORAL FORMULAS



Here, not every (infinite) path from the start-node $\Diamond A$ represents a situation that makes $\Diamond A$ true. However, each situation that makes $\Diamond A$ true corresponds to a path from the start-node.

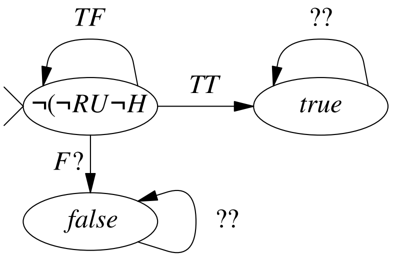


Here, every path from the start-node $\Box A$ which avoids the false-node represents a situation that makes $\Box A$ true. The converse is also true.



The truth-value pair XY means $A = X$ and $B = Y$; "?" for X or Y means "don't care".

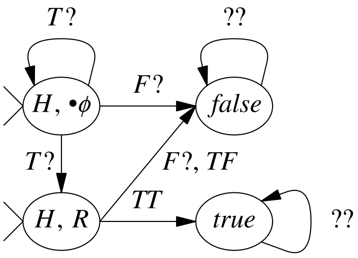
Here, every path from the node AUB which avoids the false-node represents a situation that makes AUB true. The converse is also true.



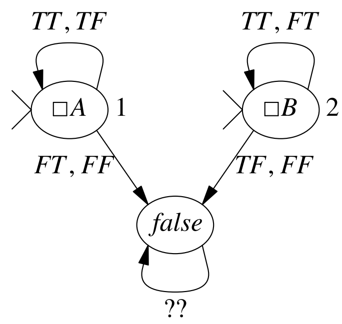
The truth-value pair XY means $H = X$ and $R = Y$.

Here, not every path from the start-node which avoids the false-node represents a situation that makes $\neg(\neg RU\neg H)$ true. However, each situation that makes $\neg(\neg RU\neg H)$ true corresponds to a path from the start-node.

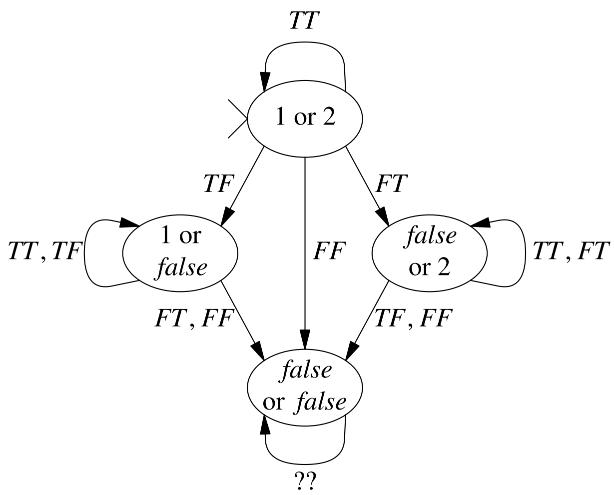
Rewriting $\phi = \neg(\neg RU\neg H) = \neg[\neg H \vee (\neg R \wedge \bullet(\neg RU\neg H))] \quad = H \wedge (R \vee \bullet\neg(\neg RU\neg H))] = [H\wedge R] \vee [H\wedge\bullet\phi]$ gives the following, which becomes the same as above after converting to deterministic form.



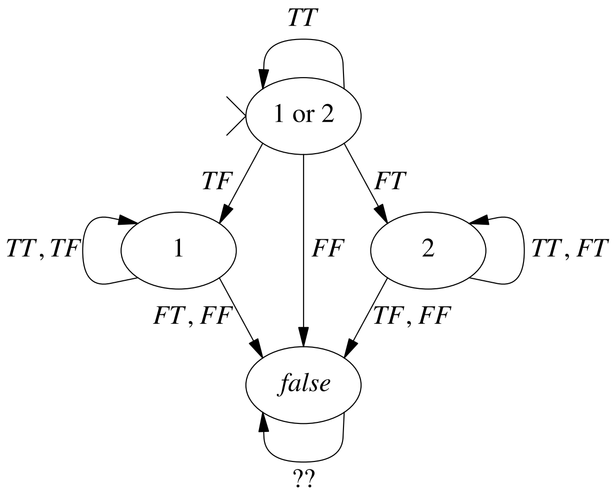
STATE-GRAPH FOR $(\Box A \vee \Box B)$



Non-deterministic form



Deterministic form

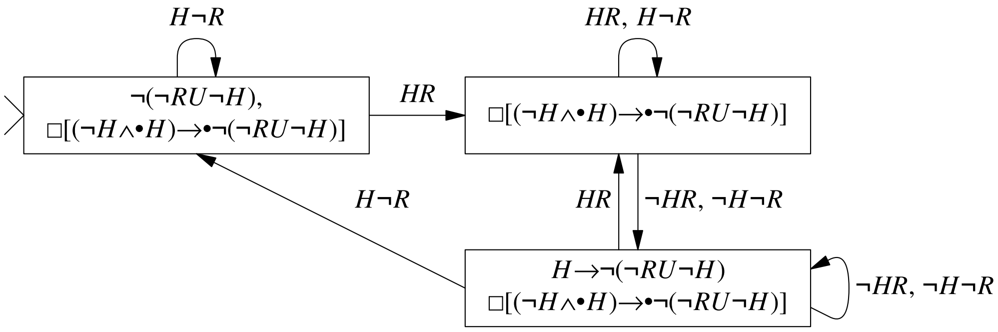


After simplifying state-names

Here, every path from the start-node which avoids the false-node represents a situation that makes AUB true. The converse is also true.

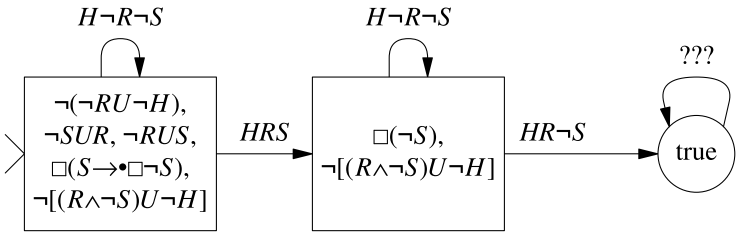
SOME MORE EXAMPLES OF STATE-GRAPH

The state-graph for "It rains at least once during the hurrican which occurs now".



Not shown is the false-node.

The state-graph for "It rains at least twice during the hurrican which occurs now".



NORMAL FORMS OF A TEMPORAL FORMULA

•-formula:	a formula of the form $\bullet\phi$
♦-formula:	a formula of the form $\blacklozenge\phi$
□-formula:	a formula of the form $\Box\phi$
U-formula:	a formula of the form $\phi U \psi$

Normal form: A formula with the properties (1)-(5) below.

(1) ¬-operators apply only to propositions.
(2) •-operator are moved as far outward as possible.
(3) □-operator is not applied to a conjunction or to another □-formula.
(4) ♦-operator is not applied to a disjunction or to another ♦-formula.
(5) The first argument of an U-formula is not a conjunction and the second argument is not a disjunction.

Normal form of a formula may not be unique. (Any normal form will suffice for construction of state-graph.)

Example. $\blacklozenge(p \wedge (q \vee r)) = \blacklozenge[(p \wedge q) \vee (p \wedge r)] = [\blacklozenge(p \wedge q)] \vee [\blacklozenge(p \wedge r)]$.

$\Box(p \vee (q \wedge r)) = \Box[(p \vee q) \wedge (p \vee r)] = [\Box(p \vee q)] \wedge [\Box(p \vee r)]$.

Transformation rules: replace the formula on the left of "=" by that on the right.

(1)	$\neg(\phi \vee \psi) = \neg\phi \wedge \neg\psi$	(3)	$\Box(\phi \wedge \psi) = (\Box\phi) \wedge (\Box\psi)$
	$\neg(\phi \wedge \psi) = \neg\phi \vee \neg\psi$		$\Box\Box\phi = \Box\phi$
	$\neg\bullet\phi = \bullet\neg\phi$		
	$\neg\Box\phi = \blacklozenge\neg\phi$	(4)	$\blacklozenge(\phi \vee \psi) = (\blacklozenge\phi) \vee (\blacklozenge\psi)$
	$\neg\blacklozenge\phi = \Box\neg\phi$		$\blacklozenge\blacklozenge\phi = \blacklozenge\phi$
	$\neg(\phi \,\mathrm{U}\, \psi) = \neg\psi \wedge \blacklozenge\neg\phi \wedge ((\bullet\neg\psi) \,\mathrm{U}\, (\neg\phi))$		
	$\neg\neg\phi = \phi$	(5)	$(\phi \wedge \psi) \,\mathrm{U}\, \zeta = (\phi \,\mathrm{U}\, \zeta) \wedge (\psi \,\mathrm{U}\, \zeta)$
			$\phi \,\mathrm{U}\, (\psi \vee \zeta) = (\phi \,\mathrm{U}\, \psi) \vee (\phi \,\mathrm{U}\, \zeta)$
(2)	$\neg\bullet\phi = \bullet\neg\phi$		
	$\Box\bullet\phi = \bullet\Box\phi$	(6)	$\phi \wedge (\psi \vee \zeta) = (\phi \wedge \psi) \vee (\phi \wedge \zeta)$
	$\blacklozenge\bullet\phi = \bullet\blacklozenge\phi$		$\phi \vee (\psi \wedge \zeta) = (\phi \vee \psi) \wedge (\phi \vee \zeta)$
	$(\bullet\phi) \,\mathrm{U}\, (\bullet\psi) = \bullet(\phi \,\mathrm{U}\, \psi)$		
	$(\bullet\phi) \wedge (\bullet\psi) = \bullet(\phi \wedge \psi)$		
	$(\bullet\phi) \vee (\bullet\psi) = \bullet(\phi \vee \psi)$		

Some other normalization rules:

$$\blacklozenge\Box\blacklozenge\phi = \Box\blacklozenge\phi$$
$$\Box\blacklozenge\Box\phi = \blacklozenge\Box\phi$$
$$\blacklozenge(\phi \,\mathrm{U}\, \psi) = (\blacklozenge\psi) \vee \blacklozenge\Box\phi$$
$$\Box(\phi \,\mathrm{U}\, \psi) = \Box(\phi \vee \psi)$$

RULES FOR SIMPLIFYING A FORMULA FOR CURRENT VALUES OF TEMPORAL PROPOSITIONS

Decomposition into "current" and "future" components:

$$\begin{aligned} \Box\psi &= \psi \wedge \bullet\Box\psi \\ \blacklozenge\,\psi &= \psi \vee \bullet\blacklozenge\,\psi \\ \psi \,\mathrm{U}\, \zeta &= \zeta \vee (\psi \wedge \bullet(\psi \,\mathrm{U}\, \zeta)) \end{aligned}$$

Example:

$$\begin{aligned} \Box\blacklozenge\, p &= \blacklozenge\, p \wedge \bullet\Box\blacklozenge\, p \\ &= (p \vee \bullet\blacklozenge\, p) \wedge \bullet\Box\blacklozenge\, p \end{aligned}$$

$$\begin{aligned} \blacklozenge(p\ U\ q) &= (p\ U\ q) \vee \bullet\blacklozenge(p\ U\ q) \\ &= q \vee (p \wedge \bullet(p\ U\ q)) \vee \bullet\blacklozenge(p\ U\ q) \end{aligned}$$

Simplification Rules:

v = a vector of T/F values for the propositions.

$$\begin{aligned} sim(p,\, v) &= v(p), \text{ the truth value of } p \text{ in } v \\ sim(\neg p,\, v) &= \neg v(p) \\ sim(\psi \wedge \zeta,\, v) &= sim(\psi,\, v) \wedge sim(\zeta,\, v) \\ sim(\psi \vee \zeta,\, v) &= sim(\psi,\, v) \vee sim(\zeta,\, v) \\ sim(\bullet\psi,\, v) &= \psi \\ sim(\Box\psi,\, v) &= sim(\psi,\, v) \wedge \Box\psi \\ sim(\blacklozenge\psi,\, v) &= sim(\psi,\, v) \vee \blacklozenge\psi \\ sim(\psi U \zeta,\, v) &= sim(\zeta,\, v) \vee [sim(\psi,\, v) \wedge (\psi U \zeta,\, v)] \end{aligned}$$

CONSTRUCTION OF THE STATE-GRAPH FOR CHECKING SATISFIABILITY OF A TEMPORAL FORMULA

Simple normal form formula:

A normal form formula which is not a disjunction or conjunction of smaller normal form formulas.

Converting a set of normal form formulas to sets of simple normal form formulas:

(1)	$\{\psi_1 \wedge \psi_2, \psi_3, \cdots, \psi_k\}$	$= \{\psi_1, \psi_2, \psi_3, \cdots, \psi_k\}$
(2)	$\{\psi_1 \vee \psi_2, \psi_3, \cdots, \psi_k\}$	$= \{\psi_1, \psi_3, \cdots, \psi_k\}$ and $\{\psi_2, \psi_3, \cdots, \psi_k\}$.
(3.1)	$\{\blacklozenge\psi_1, \blacklozenge(\psi_1 \wedge \psi_2), \psi_3, \cdots, \psi_k\}$	$= \{\blacklozenge(\psi_1 \wedge \psi_2), \psi_3, \cdots, \psi_k\}$
(3.2)	$\{\psi_1, \blacklozenge\psi_1, \psi_3, \cdots, \psi_k\}$	$= \{\psi_1, \psi_3, \cdots, \psi_k\}$
(4.1)	$\{\Box\psi_1, \Box(\psi_1 \vee \psi_2), \psi_3, \cdots, \psi_k\}$	$= \{\Box\psi_1, \psi_3, \cdots, \psi_k\}$
(4.2)	$\{\psi_1, \Box\psi_1, \psi_3, \cdots, \psi_k\}$	$= \{\Box\psi_1, \psi_3, \cdots, \psi_k\}$, if ψ_1 is not a $\blacklozenge$-formula

(Remove duplicates, if any, for each set on the right.)

Algorithm DIGRAPH:

Input: A temporal formula ϕ.
Output: The satisfiability state-graph $G = G(\phi)$.

1. Normalize ϕ and create one or more sets S_j of simple normal form formulas.
2. Initialize the digraph G with one node for each S_j; mark each of these nodes "unexpanded". Let n = #(distinct propositions in the formulas in ϕ).
3. If there is an unexpanded node in G, choose such a node Q; otherwise stop.
4. For each of 2^n possible T/F-vectors $v = (v_1, v_2, \cdots, v_n)$, v_i = value of temporal proposition p_i, do the following:
   (a) If $Q = \{\text{true}\}$ or $Q = \{\text{false}\}$, then add the arc (Q, v, Q), i.e., the arc from Q to Q with the label v; otherwise, do the Steps (b)-(d).
   (b) Suppose $Q = \{\phi_1, \phi_2, \cdots, \phi_k\}$ and $\psi_i = sim(\phi_i, v)$, $1 \le i \le k$.
   (c) If some ψ_i = F, then let $m = 1$ and let $Q_1 = \{\text{false}\}$; on the other hand, if each ψ_i = T, then let m = 1 and $Q_1 = \{\text{true}\}$. Otherwise, let $Q' = \{\psi_1, \psi_2, \cdots, \psi_j\}$, $0 < j \le k$, be the set of those ψ_i's which are different from T. Let $Q_1, Q_2, \cdots, Q_m$ $(m \ge 1)$ be the sets of simple formulas obtained from Q'.
   (d) For each Q_j, $1 \le j \le m$, do the following.
      (i) If Q_j is not an existing node in G, then create such a node and mark it "unexpanded".
      (ii) Add the arc (Q, v, Q_j).
5. Mark the node Q "expanded", and goto Step (3).

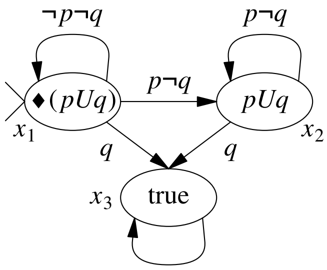
FINITE SUPPORT OF ♦-FORMULA IN A SATISFIABILITY-DIGRAPH

Finite Support of $\blacklozenge\phi$ at a node x:

- The labeled arc (y, v, z), where y may be different from x, is a *finite support* if there is a path $\pi(x, y)$ from x to y such that for labels v' on the arcs in $\pi(x, y)$ we have $sim(\phi, v') = F$ and
- $sim(\phi, v) = T$ (and hence $sim(\blacklozenge\phi, v) = T$).

That is, (y, v, z) is the first arc on a path from x at which ϕ becomes true.

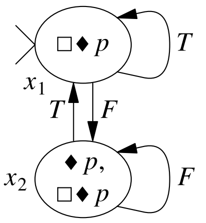
Example. The arcs (x_1, v, x_3) and (x_2, v, x_3) are finite supports for $\blacklozenge(pUq)$ at x_1 for each $v = pq, \neg pq$.



$$\begin{aligned}\blacklozenge(pUq) &= (pUq) \vee \bullet\blacklozenge(pUq)\\ &= q \vee (p\wedge\bullet(pUq)) \vee \bullet\blacklozenge(pUq)\end{aligned}$$

$$\begin{aligned}sim(\blacklozenge(pUq), TF) &= (pUq) \vee \blacklozenge(pUq)\\ &= (pUq)\end{aligned}$$

Example. The arcs (x_2, T, x_1) is the finite support for $\blacklozenge p$ at x_2. (Note that $\blacklozenge p$ is not eliminated in node x_2, which would cause node x_2 to be merged with node x_1, the issue of support will go away. However, not all paths in the diagram give situations which make $\Box\blacklozenge p$ true.)



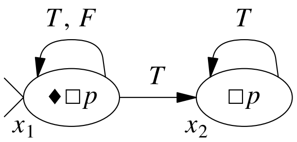
$$\begin{aligned}\Box\blacklozenge p &= \blacklozenge p \wedge \bullet\Box\blacklozenge p\\ &= (p \vee \bullet\blacklozenge p) \wedge \bullet\Box\blacklozenge p\end{aligned}$$

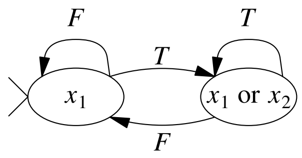
$$sim(\Box\blacklozenge p, F) = \blacklozenge p \wedge \Box\blacklozenge p$$

Each path π (from the start-node) in which every $\blacklozenge$-formula in every node in π is supported by some arc in π makes the formulas in the start-node true.

EXERCISE

1. Shown below are the non-deterministic satisfibility state-graph for $\blacklozenge\Box p$ and its deterministic form. In what sense we can say that the arc (x_2, T, x_2) is a finite support of the $\blacklozenge$-formula in x_1 (since the only paths that makes $\blacklozenge\Box p$ true are those which contains that arc)? Is there is a finite support for $\blacklozenge\Box p$ at x_1 in the deterministic form?





$\blacklozenge\Box p = \Box p \vee \bullet\blacklozenge\Box p = (p \wedge \bullet\Box p) \vee \bullet\blacklozenge\Box p$, $sim(\blacklozenge\Box p, T) = \Box p \vee \blacklozenge\Box p$.

ULTIMATELY PERIODIC SOLUTION

Periodic solution (with a period n):

- For each temporal proposition P, $P(i+n) = P(i)$ for all $i \geq 1$.
- The smallest n is called *the period.*

Example 1. It rains during each hurricane.

$H =$ (F, T, F, F, T, T, F, F, F, T, T, T, ⋯) A non-periodic solution
$R =$ (F, T, F, T, F, T, F, T, F, T, F, T, ⋯) although R is periodic

$H =$ (F, T, T, F, T, T, F, T, T, F, ⋯) A periodic solution
$R =$ (F, T, F, T, F, T, F, T, F, T, ⋯) (period = 6 = 2×3)

Example 2. $\phi = \neg A \wedge \bullet\neg A \wedge \bullet\bullet\Box A$ has no periodic solution.

$A =$ (F, F, T, T, T, T, ⋯) A non-periodic solution
$A =$ (F, F, F, T, T, T, ⋯) Another non-periodic solution

Ultimately periodic solution:

- For each temporal proposition P, $P(i+n) = P(i)$ for $i \geq m$.
  The initial non-periodic part is at most of length $(m-1)$: $P(1)$, $P(2)$, ⋯, $P(m-1)$.

Example 3. All solutions of ϕ in Example 2 are ultimately periodic, although it has no periodic solution.

If a formula ϕ is satisfiable, then it has an ultimately periodic solution.

Example 4. The statement σ = "Each run of H is finite (i.e., $\neg H$ occurs infinitely often) and is longer than its previous run" *cannot* be expressed in temporal logic because it has no ultimately periodic solution.

H = (T F T T F T T T F ⋯) makes σ true.