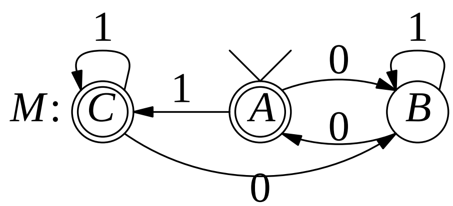
EQUIVALENCE OF STATES
AND STATE-MINIMIZATION OF AN FSA

The "future" Language of a State:

- $L_{q_i}(M) = \{x\colon x$ takes M from q_i to a final-state$\}$; in short, L_{q_i}.
- $L_{q_i}(M)$ captures the notion "future" of state q_i.
- $L_{start-state}(M) = L(M)$.

Example. $L_B(M) = \{0, 01, 10, 000, 011, 101, 110, \cdots\} = L_{0\text{-}odd}$



Equivalence of Two States q_i and q_j of an FSA:

- q_i is equivalent to q_j (in short, $q_i \approx q_j$) if $L_{q_i} = L_{q_j}$.

Question:

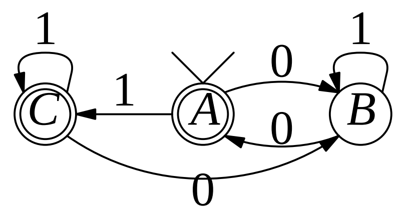
•? How to determine the equivalence of q_i and q_j in an FSA M? (Here, the problem is that $L_{q_i}(M)$ and $L_{q_j}(M)$ can be infinite!)

***m*-equivalence:** A finite form of "$\approx$"

- $L_{q_i}^{(m)} = \{x\colon x \in L_{q_i}$ and $|x| = m\}$, a finite subset of $\subseteq L_{q_i}$.
  - $L_{q_i}(0) = \{\lambda\}$, if q_i is a final-state; otherwise, $= \varnothing$.
  - $L_{q_i}^{(k)} \cap L_{q_i}^{(m)} = \varnothing$ for $k \neq m$ and $\bigcup_{m \geq 0} L_{q_i}^{(m)} = L_{q_i}$.
- q_i is m-equivalent to q_j (in short, $q_i \approx_m q_j$) if $L_{q_i}^{(m)} = L_{q_j}^{(m)}$.

Question: What is $L_B^{(3)}$ for the FSA above?

m-EQUIVALENCE



$L_{q_i}^{(m)}$	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$\cdots$
$q_i = A$	$\{\lambda\}$	$\{1\}$	$\{00, 11\}$	001, 010, 100, 111	$\cdots$
$q_i = B$	$\varnothing$	$\{0\}$	$\{01, 10\}$	000, 011, 101, 110	$\cdots$
$q_i = C$	$\{\lambda\}$	$\{1\}$	$\{00, 11\}$	001, 010, 100, 111	$\cdots$

(1) The states A and C are m-equivalent for each $m \geq 0$.
(2) The states A and B are not m-equivalent for any $m \geq 0$.

0-equivalence: For any FSA,

- All final states are 0-equivalent to each other, and all non-final states are 0-equivalent to each other.

STATE-MINIMIZATION ALGORITHM

Algorithm-1 (using $L_{q_i}^{(m)}$):

1. Begin with two groups of states: F (the final-states) and $Q - F$ (the non-final states), corresponding to the 0-equivalence classes.
2. For each $m = 1, 2, \cdots$ do the following until you reach a stage where no state-group is split up.

   Decompose each state-group into two or more groups, if necessary, based on $L_{q_i}^{(m)}$, i.e., if q_i and q_j are presently in the same group and $L_{q_i}^{(m)} \neq L_{q_j}^{(m)}$, then separate them into different groups. (At this point, q_i and q_j are in the same group if and only if $q_i \approx_k$ for all $k \leq m$.)
3. The minimum-state FSA is obtained by merging the states in each group into a single state.

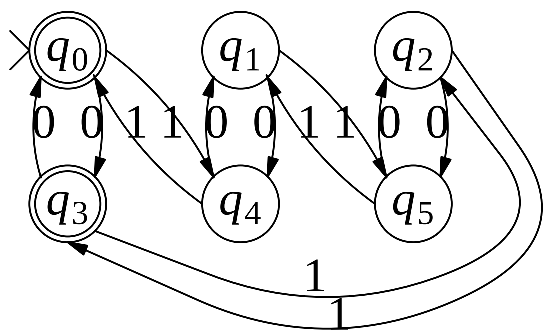
Algorithm-2 (*without* using $L_{q_i}^{(m)}$):

1. Begin with two groups of states: F (the final-states) and $Q - F$ (the non-final states).
2. For each $m = 1, 2, \cdots$ do step (3) until you reach a stage where no group is split up.
3. Repeat the following for each input symbol $a_k \in Z$.
   - (i) Compute $\delta(q_i, a_k)$ for all states.
   - (ii) Decompose each group into two or more groups, if necessary, based on $\delta(q_i, a_k)$ as follows:

     If q_i and q_j are presently in the same group and $\delta(q_i, a_k)$ and $\delta(q_j, a_k)$ are not in the same group, then separate them into two different groups.
4. The minimum-state FSA is obtained by merging the states in each group into a single state.

ILLUSTRATION OF ALGORITHM-1

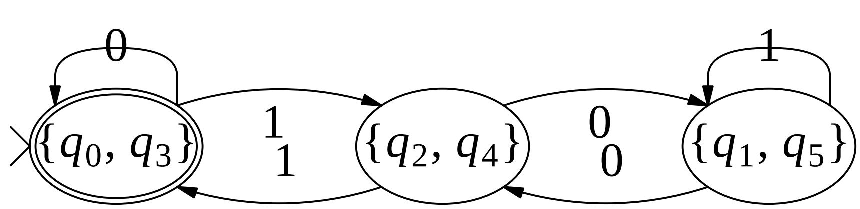
State (p, r): p = odd/even parity of number of binary bits read and r = remainder(n_j, 3), where n_j = binary number read.



q_0 = (even, 0)
q_1 = (even, 1)
q_2 = (even, 2)
and so on q_5 = (odd, 2)

State	$Lq_i^{(m)}$			
	$m = 0$	$m = 1$	$m = 2$	$m = 3$
q_0	$\{\lambda\}$	$\{0\}$	$\{00, 11\}$	$\{000, 011, 110\}$
q_3	$\{\lambda\}$	$\{0\}$	$\{00, 11\}$	$\{000, 011, 110\}$
q_1	$\varnothing$	$\varnothing$	$\{01\}$	$\{010, 101\}$
q_2	$\varnothing$	$\{1\}$	$\{10\}$	$\{001, 100, 111\}$
q_4	$\varnothing$	$\{1\}$	$\{10\}$	$\{001, 100, 111\}$
q_5	$\varnothing$	$\varnothing$	$\{01\}$	$\{010, 101\}$
	$\{q_0, q_3\}$ $\{q_1, q_2, q_3, q_4\}$	$\{q_0, q_3\}$ $\{q_1, q_5\}$ $\{q_2, q_4\}$	No change in any group.	No need to consider this.

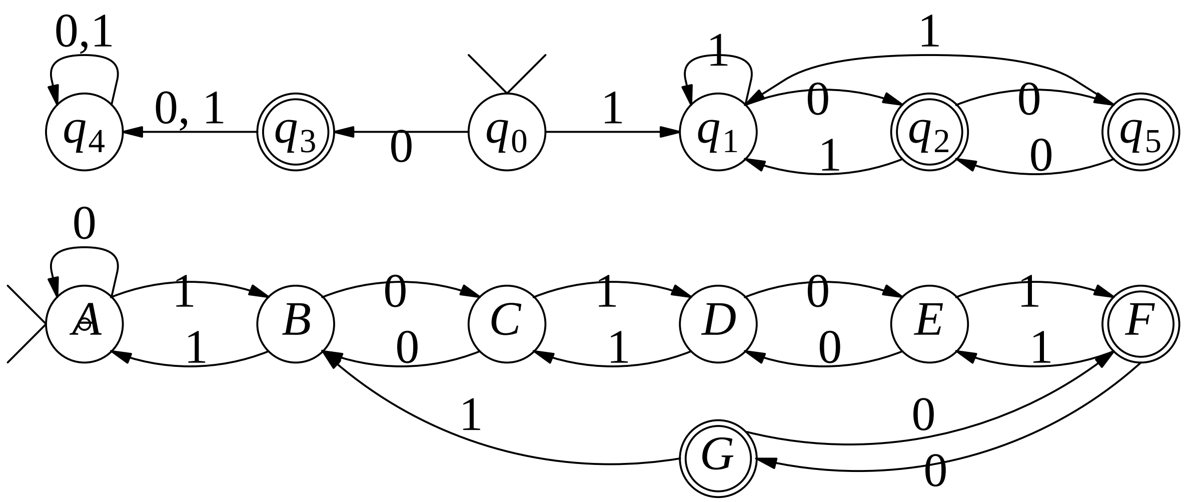
After Merging Equivalent Classes of States:



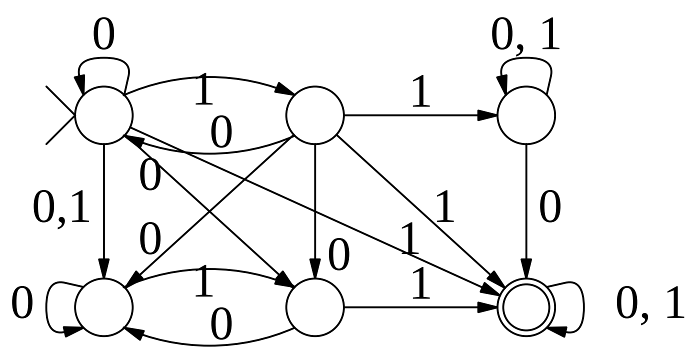
The minimum-state form of the above FSA.

EXERCISE

1. Apply Algorithm-1 to each FSA below and show all details.



2. Apply the minimization algorithm to the NFSA below (which has some important connection with $M_{has\text{-}11}$).



3. Is it true that all error states in an arbitrary FSA will be merged into a single state in the minimization process (why)?

4. Give an example FSA to show that all unreachable states in an arbitrary FSA may not get merged into a single state.

5. First, use the table on page 4 to verify the following formula for $m = 3$ and the state $q = q_0$: $L_q^{(m)} = \cup a_j L_{\delta(q,\, a_j)}^{(m-1)}$ (union over all $a_j \in \Sigma$). Then, prove that the formula holds for FSA. Finally, argue that $q \approx q'$ implies $\delta(q, a) \approx \delta(q', a)$ for all $a \in \Sigma$.

WHAT IS A STATE OF AN FSA

Importance of Answering This Question:

- It explains why certain languages like L_{sym} and $L_{a^n b^n}$ are are not acceptable by any FSA,
  - There is an FSA for a language L, i.e., L is regular if and only if it has only a finitely many quotient languagess.

States of the optimal FSA for L are the quotient languages of L.

  - The optimal FSA (which has the smallest number of states) for a language is unique; its states correspond to the quotient languages of L.

QUOTIENT LANGUAGES

Quotient of L:

- For any string x, $L/x = \{y\colon\ xy \in L\}$. Here, y are the "futures" (to do) after x.
- $L/\lambda = L$, for any L and $(L/x)/y = L/(xy)$.

Example.

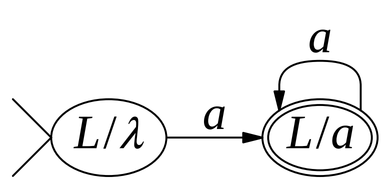
- For $L = \{a, ab, aab, b\}$,

$L/a = \{\lambda, b, ab\}$	$L/ab = \{\lambda\}$
$L/b = \{\lambda\}$	$L/ba = \emptyset$

- For $L = \{a, aa, aaa, \cdots\} = a^+$,

$L/a = \{\lambda, a, aa, \cdots\} = a^* \supset L$	$L/aa = L/a$
$L/b = \emptyset$	$L/ab = \emptyset$

States of $M(a^+)$ vs. Quotients of a^+:



Question:

•? For $L = \{a, ab, aab, b\}$, show the states $M(L)$ in terms of the quotients of L.

REGULAR LANGUAGES AND QUOTIENT LANGUAGES

$\boxed{L \text{ is regular} \Leftrightarrow L \text{ has finitely many quotient languages.}}$

Example. Let $L = \{a^n b^n : n \ge 1\} = \{ab, aabb, aaabbb, \cdots\}$.

$$\begin{array}{rll}
L/\lambda & = L & \\
L/a & = \{b, abb, aabbb, \cdots\} & = \{a^n b^{n+1} : n \ge 0\} \\
L/aa & = (L/a)/a = \{bb, abbb, \cdots\} & = \{a^n b^{n+2} : n \ge 0\} \\
L/aaa & = (L/aa)/a = \{bbb, abbbb, \cdots\} & = \{a^n b^{n+3} : n \ge 0\} \\
& \cdots &
\end{array}$$

- These quotient-languages are distinct (they differ in the smallest string in them).
- The quotients $L/b = L/bb = \cdots = \emptyset$, and they are different from the ones above. (Are there still other quotients of L?)

EXERCISE

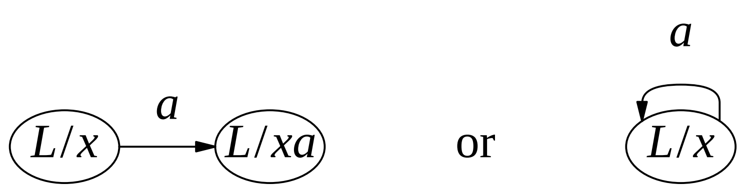
1. Find the quotient languages of $L_{has\text{-}11}$.
2. Let $L = \{a^m b^n : m, n \ge 1\} \subseteq (a+b)^*$. Give a rgular expression for each of the distinct quotient languages of L.
3. Give a clear argument to show that $L_{sym} = \{x \in (a+b)^+ : x$ is symmetric, i.e., $x = x^r\} = \{\lambda, a, b, aa, bb, aaa, aba, bab, bbb, \cdots\}$ has infinitely many quotient languages.
4. Let $L_{left\text{-}shift} = \{x \in \{b_0, b_1, b_2, b_3\}^*$: the lower part of x is the left-shift of the top part$\}$. Explain whether $L_{left\text{-}shift}$ is regular or not.

$$x = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = b_2 b_1 b_3 b_2 \in L_{left\text{-}shift}.$$

5. If L_1 and L_2 are two languages over the alphabet Σ and $L_1/x = L_2/x$ for all $x \in \Sigma^+$, then prove that $L_1 = L_2$.

UNIQUENESS OF MINIMUM FSA FOR A REGULAR LANGUAGE L

- States have the form L/x.
- Start-state = $L/\lambda = L$.
- The state L/x is final if $x \in L$, i.e., L/x contains λ.
- The transitions are of the form shown below for each $a \in \Sigma$:



(i) $L/x \neq L/xa$. (ii) $L/x = L/xa$.

- The state L/x is an error-state (dead-state) if $L/x = \emptyset$.