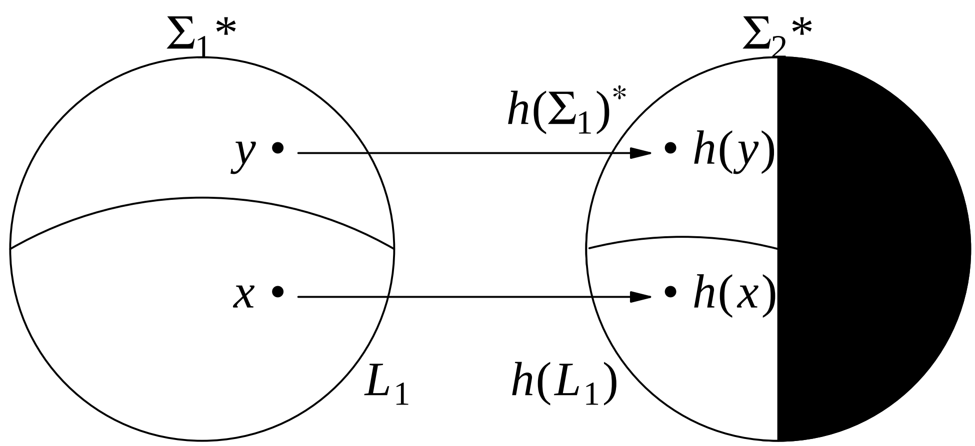
HOMOMORPHISM:
A SPECIAL FORM OF ENCODING

Homomorphism:

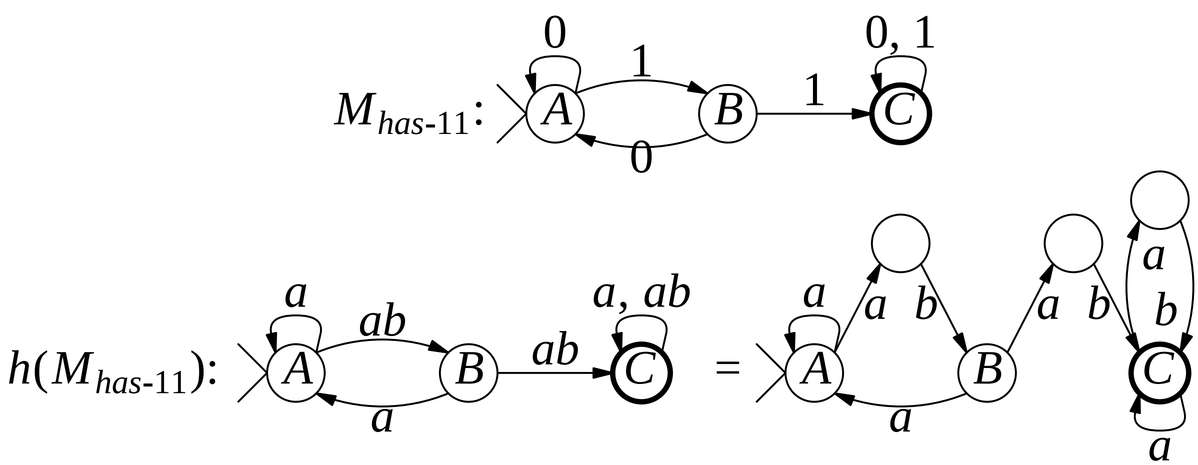
- A particular kind of mapping from languages to languages, which is based on a mapping of individual symbols.
- For a given mapping h: $\Sigma_1 \rightarrow \Sigma_2^*$ and a language $L_1 \subseteq \Sigma_1^*$,
- we extend h: $\Sigma_1^* \rightarrow \Sigma_2^*$ as follows:
  - For $x = a_1 a_2 \cdots a_k \in \Sigma_1^*$, $h(x) = h(a_1)h(a_2)\cdots h(a_k)$; $h\lambda) = \lambda$.
  - Then, $h(L_1) = \{h(x)$: $x \in L_1\}$; $h(x)$ is encoding of x.



Example. For $\Sigma_1* = \{0, 1\}$, $h(0) = a$, $h(1) = ab$, and $L_1 = L_{has\text{-}11} = \{11, 011, 110, 111, 0011, \cdots\}$.

$$h(L_1) = \{abab, aabab, ababa, ababab, aaabab, \cdots\}.$$

TRANSFORMING AN FSA BY A HOMOMORRPHISM



- The label a_i on each transition is replaced by $h(a_i)$.
- If $h(a_i) = \lambda$, then it gives a λ-transition.
- $h(M)$ maybe non-deterministic, when the transition-diagram is converted to an FSA by addition of new states.

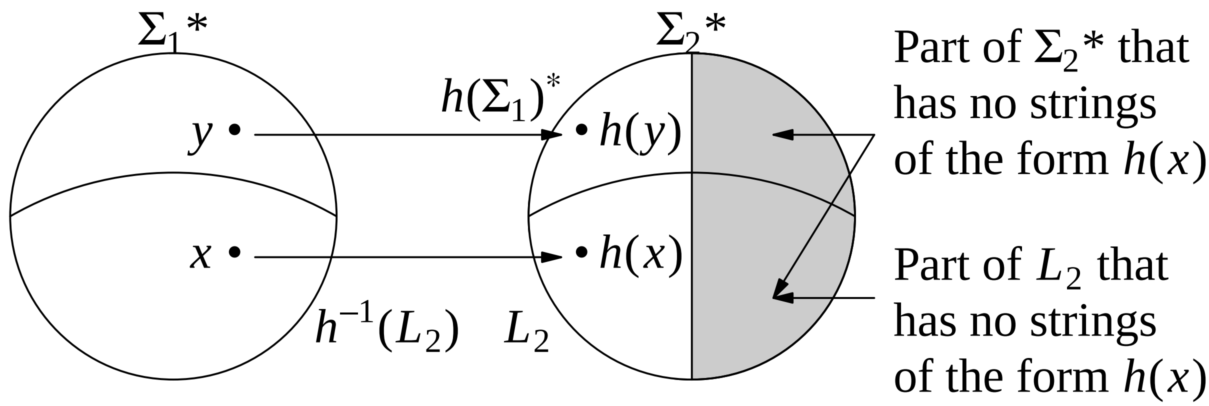
> If L_1 is regular, then $h(L_1)$ is regular for any h.

EXERCISE

1. Let $L_{verify\text{-}bin\text{-}mult\text{-}by\text{-}3} \subseteq (b_0 + b_1 + b_2 + b_3)^*$ be the language where the bottom bit-string for an input x represents the multiplication by 3 of the top bit-string corresponding to x. First, obtain an FSA (with three states) for the reverse language of $L_{verify\text{-}bin\text{-}mult\text{-}by\text{-}3}$, and then obtain an FSA for $L_{verify\text{-}bin\text{-}mult\text{-}by\text{-}3}$ itself. Finally, find an homomorphism h such that $h(L_{verify\text{-}bin\text{-}mult\text{-}by\text{-}3}) = L(M_{3n})$; here, both languages contains λ. Apply the construction of $h(M)$ to $M_{verify\text{-}bin\text{-}mult\text{-}by\text{-}3}$.
2. Is there a homomorphism h such that $h(L_{0\text{-}even}) = (0 + 1)^*$?

INVERSE HOMOMORPHISM AND $h^{-1}(M)$

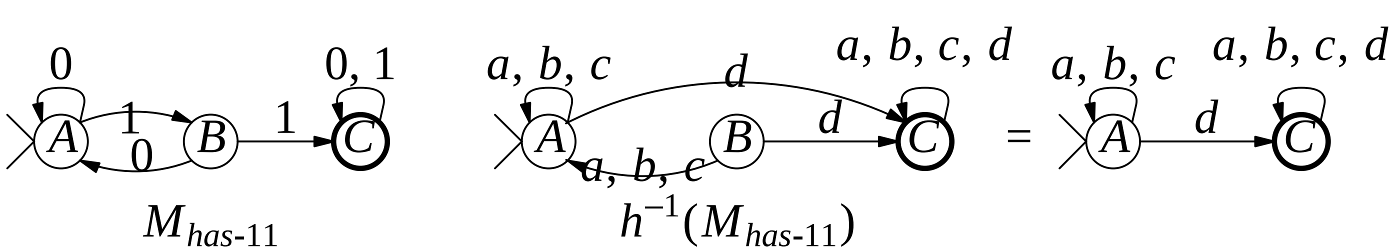
- For $z \in \Sigma_2^*$, $h^{-1}(z) = \{x \colon h(x) = z\}$; $h^{-1}(z)$ maybe $\varnothing$.
- For $L_2 \subseteq \Sigma_2^*$, $h^{-1}(L_2) = \{x \colon h(x) \in L_2\} = \bigcup_{z \in L_2} h^{-1}(z)$.



Example. For $\Sigma_1 = \{a, b, c, d\}$, $\Sigma_2 = \{0, 1\}$, and $h(a) = 0 = h(b)$, $h(c) = 00$, and $h(d) = 11$:

$h^{-1}(\lambda) = \{\lambda\}$	$h^{-1}(00) = \{aa, ab, ba, bb, c\}$
$h^{-1}(0) = \{a, b\}$	$h^{-1}(01) = \varnothing = h^{-1}(10)$
$h^{-1}(1) = \varnothing$	$h^{-1}(11) = \{d\}$
	$h^{-1}(000) = \{aaa, \cdots, bbb, ac, bc, ca, cb\}$

$$h^{-1}(L_{has\text{-}11}) = \{d, ad, bd, cd, da, db, aad, abd, bad, \cdots\}.$$



- The NFSA $h^{-1}(M)$ has the same start and final states as M.
- It has a transition $\delta(q_i, a_j) = q_k$ if $h(a_j)$ takes q_i in M to q_k.

If L_2 is regular, then $h^{-1}(L_2)$ is regular for any h.

EXERCISE

1. What is the relationship between L_2 and $h[h^{-1}(L_2)]$, and that between L_1 and $h^{-1}[h(L_1)]$?
2. Let $r = (10 + 0) * 11(0 + 1) *$ be a regular expression for the regular language L and let $h(0) = ab$ and $h(1) = ba$ a homomorphism. What is a regular expression for $h(L)$? How about $h^{-1}(L')$ if L' is given by the regular expression $r' = (ab + a) * bb(a + b) *$?