HOMOMORPHISM:
A SPECIAL FORM OF ENCODING

Homomorphism:

- A particular kind of mapping from languages to languages, which is based on a mapping of individual symbols.
- For a given mapping $h: \Sigma_1 \rightarrow \Sigma_2^*$ and a language $L_1 \subseteq \Sigma_1^*$, we extend $h: \Sigma_1^* \rightarrow \Sigma_2^*$ as follows:
  - For $x = a_1a_2\cdots a_k \in \Sigma_1^*$, $h(x) = h(a_1)h(a_2)\cdots h(a_k); h(\lambda) = \lambda$.
  - Then, $h(L_1) = \{h(x): x \in L_1\}$; $h(x)$ is encoding of $x$.

Example. For $\Sigma_1^* = \{0, 1\}$, $h(0) = a$, $h(1) = ab$, and $L_1 = L_{has-11} = \{11, 011, 110, 111, 0011, \cdots\}$.

\[ h(L_1) = \{abab, aabab, ababa, ababab, aaabab, \cdots\}. \]
TRANSFORMING AN FSA BY A HOMOMORPHISM

The label $a_i$ on each transition is replaced by $h(a_i)$.
If $h(a_i) = \lambda$, then it gives a $\lambda$-transition.
$h(M)$ maybe non-deterministic, when the transition-diagram is converted to an FSA by addition of new states.

If $L_1$ is regular, then $h(L_1)$ is regular for any $h$.

EXERCISE

1. Let $L_{verify-bin-mult-by-3} \subseteq (b_0 + b_1 + b_2 + b_3)^*$ be the language where the bottom bit-string for an input $x$ represents the multiplication by 3 of the top bit-string corresponding to $x$. First, obtain an FSA (with three states) for the reverse language of $L_{verify-bin-mult-by-3}$, and then obtain an FSA for $L_{verify-bin-mult-by-3}$ itself. Finally, find an homomorphism $h$ such that $h(L_{verify-bin-mult-by-3}) = L(M_{3n})$; here, both languages contains $\lambda$. Apply the construction of $h(M)$ to $M_{verify-bin-mult-by-3}$.

2. Is there a homomorphism $h$ such that $h(L_{0-even}) = (0 + 1)^*$?
INVERSE HOMOMOMORPHISM AND $h^{-1}(M)$

- For $z \in \Sigma_2^*$, $h^{-1}(z) = \{ x : h(x) = z \}$; $h^{-1}(z)$ may be $\emptyset$.
- For $L_2 \subseteq \Sigma_2^*$, $h^{-1}(L_2) = \{ x : h(x) \in L_2 \} = \bigcup_{z \in L_2} h^{-1}(z)$. 

Example. For $\Sigma_1 = \{a, b, c, d\}$, $\Sigma_2 = \{0, 1\}$, and $h(a) = 0 = h(b)$, $h(c) = 00$, and $h(d) = 11$:

- $h^{-1}(\lambda) = \{ \lambda \}$
- $h^{-1}(00) = \{ aa, ab, ba, bb, c \}$
- $h^{-1}(0) = \{ a, b \}$
- $h^{-1}(01) = \emptyset = h^{-1}(10)$
- $h^{-1}(1) = \emptyset$
- $h^{-1}(11) = \{ d \}$
- $h^{-1}(000) = \{ aaa, \ldots, bbb, ac, bc, ca, cb \}$

$h^{-1}(L_{has-11}) = \{ d, ad, bd, cd, da, db, aad, abd, bad, \ldots \}$. 

- The NFSA $h^{-1}(M)$ has the same start and final states as $M$.
- It has a transition $\delta(q_i, a_j) = q_k$ if $h(a_j)$ takes $q_i$ in $M$ to $q_k$.

If $L_2$ is regular, then $h^{-1}(L_2)$ is regular for any $h$. 
EXERCISE

1. What is the relationship between $L_2$ and $h[h^{-1}(L_2)]$, and that between $L_1$ and $h^{-1}[h(L_1)]$?

2. Let $r = (10 + 0) * 11(0 + 1) *$ be a regular expression for the regular language $L$ and let $h(0) = ab$ and $h(1) = ba$ a homomorphism. What is a regular expression for $h(L)$? How about $h^{-1}(L')$ if $L'$ is given by the regular expression $r' = (ab + a) * bb(a + b) *$?